Connections from trivializations

Dedicated to Professor Ivan Kolár on the occasion of his 80th birthday
with respect and gratitude

Abstract. Let $P$ be a principal fiber bundle with the basis $M$ and with the structural group $G$. A trivialization of $P$ is a section of $P$. It is proved that there exists only one gauge natural operator transforming trivializations of $P$ into principal connections in $P$. All gauge natural operators transforming trivializations of $P$ and torsion free classical linear connections on $M$ into classical linear connections on $P$ are completely described.

Introduction. All manifolds considered in the paper are assumed to be finite dimensional, Hausdorff, second countable, without boundary and smooth (of class $C^\infty$). Maps between manifolds are assumed to be smooth (of class $C^\infty$).

Let $M$ be a manifold and let $p : P \to M$ (or shortly $P$) be a principal fibre bundle with the basis $M$ and with the structure group $G$. Let $R : P \times G \to P$ be the right action.

A trivialization of $P$ is a section $\sigma : M \to P$ of $P$.

A principal connection in $P$ is a right invariant sub-bundle $\Gamma$ of the tangent bundle $TP$ of $P$ such that $TP = VP \oplus \Gamma$, where $VP = \bigcup_{x \in M} TP_x \subset TP$ is the vertical bundle (over $P$) of $P \to M$, see [4]. The right invariance of $\Gamma$ means that $TR_\xi(\Gamma) = \Gamma$ for any $\xi \in G$.

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Equivalently, a principal connection in $P$ is a right invariant section $\Gamma : P \to J^1P$ of the first jet prolongation $\pi_0^1 : J^1P \to P$ of $P \to M$. Then the equivalence is given by the equality $\Gamma_p = \text{im}\ T_x\sigma$, where $\Gamma(p) = j_x^1\sigma$, $p \in P_x$, $x \in M$.

The right action of $G$ on $P$ induces a right action of $G$ on the first jet prolongation $J^1P$ of $P$ by $v \cdot g = j^1_x(\sigma \cdot g)$, $v = j_x^1\sigma \in J^1P$, $g \in G$. The orbit of $j_x^1\sigma$ with respect to the action will be denoted by $[j_x^1\sigma]_G$. The fiber bundle $QP := J^1P/G = \{[j_x^1\sigma]_G \mid j_x^1\sigma \in J^1P\}$ over $M$ of orbits of the right action of $G$ on $J^1P$ is called the principal connection bundle of $P$. Principal connections $\Gamma : P \to J^1P$ in $P$ are in bijection with sections $\Gamma : M \to QP$ of $QP \to M$. The bijection is given by $\Gamma(x) := [j_x^1\sigma]_G$, where $\Gamma(p) = j_x^1\sigma$, $p \in P_x$, $x \in M$.

If $P = LM$ is the principal bundle (with the structure group $G = GL(m)$) of linear frames of a manifold $M$, a principal connection $\Lambda$ in $LM$ is called a classical linear connection on $M$.

Equivalently, a classical linear connection on $M$ is a bilinear map $\nabla = \nabla^\Lambda : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that $\nabla f X Y = f \nabla X Y$ and $\nabla X f Y = f \nabla X Y + X(f)Y$ for any vector fields $X, Y \in \mathcal{X}(M)$ on $M$ and any map $f : M \to \mathbb{R}$, see [4].

A classical linear connection $\Lambda$ on $M$ is torsion-free if its torsion tensor $T_\Lambda$ vanishes. (The torsion tensor $T_\Lambda$ is a tensor field of type $(1, 2)$ on $M$ given by $T_\Lambda(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.)

Equivalently, a classical linear connection on $M$ is a linear section $\Lambda : TM \to J^1TM$ of the first jet prolongation $J^1TM \to TM$ of the tangent bundle $TM$ of $M$, see [6].

In Section 1 of the present paper, we study the problem how a trivialization $\sigma$ of $P$ can induce a principal connection $A(\sigma)$ in $P$. This problem is reflected in the concept of gauge natural operators $A$ in the sense of [6] producing principal connections $A(\sigma) : M \to QP$ in $P \to M$ from trivializations $\sigma$ of $P$. We prove that any gauge natural operator $A$ in question is given by $A(\sigma)(x) := [j_x^1\sigma]_G$.

In Section 2 of the present paper, we study the problem how a pair $(\sigma, \Lambda)$ of a trivialization $\sigma$ of $P$ and a torsion free classical linear connection $\Lambda$ on $M$ can induce a classical linear connection $A(\sigma, \Lambda)$ on $P$. This problem is reflected in the concept of gauge natural operators $A$ in the sense of [6] producing classical linear connections $A(\sigma, \Lambda)$ on $P$ from trivializations $\sigma$ of $P$ by means of classical linear connections $\Lambda$ on $M$. We describe completely all gauge natural operators $A$ in question.

Natural operators producing connections have been studied in many papers, e.g. [1], [2], [3], [5], [6], etc.

1. Principal connections in $P$ from trivializations of $P$. Let $G$ be a Lie group and $m$ a positive integer. Let $\mathcal{PB}_m(G)$ be the category of principal bundles with $m$-dimensional bases and with the structure group $G$ and all
(local) principal bundle isomorphisms with \( \text{id}_G \) as the group isomorphism. Let \( \mathcal{FM} \) be the category of fibred manifolds and their fibred maps.

Any \( \mathcal{PB}_m(G) \)-object \( P \) over \( M \) induces the principal connection bundle \( QP \rightarrow P/G \) over \( M \) (see Introduction) and any \( \mathcal{PB}_m(G) \)-morphism \( f : P \rightarrow P' \) with the base map \( f : M \rightarrow M' \) induces fibred map \( Qf : QP \rightarrow QP' \) covering \( f \) defined by \( Qf \circ [j^1_2 \sigma]_G := [j^1_2 \sigma \circ \overline{f}^{-1}]_G, [j^1_2 \sigma]_G \in QP \). The correspondence \( Q : \mathcal{PB}_m(G) \rightarrow \mathcal{FM} \) is a gauge bundle functor in the sense of [6].

The general concept of gauge natural operators can be found in [6]. In particular, a gauge natural operator \( A : \mathcal{PB}_m(G) \rightarrow Q \) transforming trivializations of \( P \) into principal connections in \( P \) is a \( \mathcal{PB}_m(G) \)-invariant system of operators

\[
A : C_M^\infty(P) \rightarrow C_M^\infty(QP)
\]

for all \( \mathcal{PB}_m(G) \)-objects \( P \rightarrow M \), where \( C_M^\infty(P) \) is the set of all trivializations of \( P \) (possible \( C_M^\infty(P) = \emptyset \) for some \( P \)) and \( C_M^\infty(QP) \) is the set of all principal connections in \( P \). The invariance of \( A \) means that if \( \sigma \in C_M^\infty(P) \) and \( \sigma^1 \in C_M^\infty(P') \) are \( f \)-related by an \( \mathcal{PB}_m(G) \)-map \( f : P \rightarrow P' \) with the base map \( f : M \rightarrow M' \) (i.e. \( f \circ \sigma = \sigma^1 \circ \overline{f} \)), then \( A(\sigma) \) and \( A(\sigma^1) \) are \( Qf \)-related (i.e. \( Qf \circ A(\sigma) = A(\sigma^1) \circ f \)). By [6], any (gauge) natural operator \( A \) is local and it can be extended uniquely on locally defined trivializations.

**Example 1.** For any \( \mathcal{PB}_m(G) \)-object \( P \) over \( M \) we have a function

\[
D : C_M^\infty(P) \rightarrow C_M^\infty(QP), \quad D(\sigma)(x) = [j^1_0 \sigma]_G, \quad \sigma \in C_M^\infty(P), \quad x \in M.
\]

The family \( D : \text{id}_{\mathcal{PB}_m(G)} \rightarrow Q \) of functions \( D \) for \( \mathcal{PB}_m(G) \)-objects \( P \) over \( M \) is a gauge natural operator (in question).

We have the following theorem.

**Theorem 1.** The gauge natural operator \( D : \text{id}_{\mathcal{PB}_m(G)} \rightarrow Q \) (of Example 1) is the unique one, transforming trivializations of \( P \) into principal connections in \( P \).

**Proof.** Suppose that \( A : \text{id}_{\mathcal{PB}_m(G)} \rightarrow Q \) is a gauge natural operator. We have to show that \( A(\sigma)(x) = [j^1_0 \sigma]_G \) for any \( \mathcal{PB}_m(G) \)-object \( P \) over \( M \), any \( \sigma \in C_M^\infty(P) \) and any \( x \in M \).

Because of the invariance of \( A \) with respect to the principal bundle charts, we may assume that \( P = \mathbb{R}^m \times G \) (the trivial principal bundle over \( M = \mathbb{R}^m \)), \( x = 0 \in \mathbb{R}^m \) and \( \sigma(y) = (y, h(y)), y \in \mathbb{R}^m, h : \mathbb{R}^m \rightarrow G \). Then by the invariance of \( A \) with respect to the \( \mathcal{PB}_m(G) \)-morphism \( f : \mathbb{R}^m \times G \rightarrow \mathbb{R}^m \times G, f(y, \xi) = (y, h(y)^{-1} \cdot \xi) \), we may assume that \( \sigma(y) = (y, e_G) \), \( y \in \mathbb{R}^m \).

Denote \( A(\sigma)(0) = [j^1_0 \rho]_G, \rho(0) = (0, e_G) \). Using the invariance of \( A \) with respect to \( \mathcal{PB}_m(G) \)-maps \( a_t : \mathbb{R}^m \times G \rightarrow \mathbb{R}^m \times G, a_t(y, \xi) = (t^2 y, \xi), t > 0 \), we get the homogeneous condition \( A(\sigma)(0) = [j^1_0 (a_t \circ \rho \circ (a_t)^{-1})]_G \),
2. Classical linear connections on \( P \) from trivializations of \( P \to M \) by means of classical linear connections on \( M \).

Classical linear connections on a manifold \( M \) are principal connections in the principal bundle \( LM \) of linear frames on \( M \). Thus classical linear connections on \( M \) are elements from \( C_M^\infty(Q(LM)) \). We denote the set of torsion free classical linear connections on \( M \) by \( C_M^\infty(Q(LM)) \).

By [6], a gauge natural operator \( A : \text{id}_{\mathcal{PB}_m(G)} \times Q_L B \to Q_L \) transforming pairs consisting of trivializations of \( P \) and torsion free classical linear connections on \( M \) into classical linear connections on \( P \) is a \( \mathcal{PB}_m(G) \)-invariant family of regular operators

\[
A : C_M^\infty(P) \times C_M^\infty(Q_L(LM)) \to C_P^\infty(Q(LP))
\]

for \( \mathcal{PB}_m(G) \) objects \( P \) over \( M \). The regularity of \( A \) means that \( A \) transforms smoothly parametrized families of pairs of trivializations of \( P \) and torsion free classical linear connections on \( M \) into smoothly parametrized families of classical linear connections on \( P \). By [6], any (gauge) natural operator \( A \) is local and it can be extended uniquely on locally defined pairs \((\sigma, \Lambda)\) in question.

**Example 2.** Let \( P \) be an \( \mathcal{PB}_m(G) \)-object over \( M \). In Sect. 54.7 in [6], the authors construct canonically the classical linear connection \( N(D, \Lambda) \) on \( P \) from a principal connection \( D \) in \( P \) by means of a classical linear connection \( \Lambda \) on \( M \). So, using a trivialization \( \sigma \in C_M^\infty(P) \) of \( P \) and a torsion free classical linear connection \( \Lambda \) on \( M \) we can produce a classical linear connection

\[
Q(\sigma, \Lambda) := N(D(\sigma), \Lambda)
\]

on \( P \), where \( D(\sigma) \) is the principal connection in \( P \) from \( \sigma \) as in Example 1. The family \( Q : \text{id}_{\mathcal{PB}_m(G)} \times Q_L B \to Q_L \) of functions \( Q \) is a gauge natural operator (in question).

**Example 3.** Let

\[
\Delta : G \to T_{(0, e_G)}(\text{Gl}(m)) \otimes T^*_{(0, e_G)}(\text{Gl}(m)) \otimes T_{(0, e_G)}(\text{Gl}(m))
\]

be a smooth map such that \( \Delta(\xi) \) is a \( \text{Gl}(m) \times \{id_G\} \)-invariant tensor of type \((1, 2)\) on \( \text{Gl}(m) \times G \) at \((0, e_G)\) for any \( \xi \in G \). Then we have gauge natural operator

\[
A^{<\Delta>} : \text{id}_{\mathcal{PB}_m(G)} \times Q_L B \to Q_L
\]

defined as follows.

Let \( \sigma \in C_M^\infty(P) \), \( \Lambda \in C_M^\infty(Q_L(LM)) \), \( p \in P_x, x \in M \). There is a principal bundle chart \( \varphi : P_U \to \text{Gl}(m) \times G \) with \( \varphi(p) = (0, e_G) \) and sending \( \sigma|_U \) into a
constant section \( \sigma^o = (id_{R^m}, \xi^o) \in C^\infty_{R^m}(R^m \times G) \) for some \( \xi^o \in G \). Clearly, \( \xi^o \) is defined by \( \sigma(x) = R_{\xi^o}(p) \). Denote the base map of \( \varphi \) by \( \varphi : U \rightarrow R^m \).

Let \( \Lambda' \) be the image of \( \Lambda U \) by \( \varphi \) and let \( \psi \) be a \( \Lambda' \)-normal coordinate system with center 0. Replacing \( \varphi \) by \( (\psi \times id_G) \circ \varphi \), we may additionally assume that \( \varphi \) is a normal coordinate system of \( \Lambda \) with center \( x \).

Recalling that \( QLP \) is the affine bundle with \( TP \otimes T^*P \otimes T^*P \) as the corresponding vector bundle, we put

\[
A^{<\Delta>}(<\sigma, \Lambda>)(p) := Q(<\sigma, \Lambda>)(p) + T_{(0,e_G)}(\varphi^{-1} \otimes T^*_{(0,e_G)}(\varphi^{-1}(\Delta(\xi^o))),
\]

where \( Q \) is as in Example 2. If \( \varphi_1 \) is another chart, then \( \varphi_1 = (B \times id_G) \circ \varphi \) for a linear isomorphism \( B \in GL(R^m) \). So, the definition of \( A^{<\Delta>}(<\sigma, \Lambda>)(p) \) is independent of the choice of \( \varphi \) because of the invariance of \( \Delta(\xi^o) \).

We have the following theorem.

**Theorem 2.** Let \( A : id_{PB_m} \times Q, LB \rightarrow QL \) be a gauge natural operator. There is the smooth map \( \Delta : G \rightarrow T_{(0,e_G)}(R^m \times G) \otimes T^*_{(0,e_G)}(R^m \times G) \) such that \( \Delta(\xi) \) is \( GL(R^m) \times \{id_G\} \)-invariant for any \( \xi \in G \) and \( A = A^{<\Delta>} \).

The maps \( \Delta \) (in question) are in bijection with the triples \( (a,b,c) \) of smooth maps \( a, b : G \rightarrow \text{Lie}(G)^* \) and \( c : G \rightarrow \text{Lie}(G) \otimes \text{Lie}(G)^* \otimes \text{Lie}(G)^* \), where \( \text{Lie}(G) \) is the Lie algebra of \( G \). So, if we choose the basis in \( \text{Lie}(G) \), the gauge natural operators \( A \) (in question) are in bijection with the \((2k + k^3)\)-tuples of smooth maps \( G \rightarrow R \), where \( k = \dim(G) \).

**Proof.** We have to put

\[
\Delta(\xi^o) = A(<\sigma^o, \Lambda^o>)(0,e_G) - Q(<\sigma^o, \Lambda^o>)(0,e_G),
\]

where \( \xi^o \in G, \sigma^o = (id_{R^m}, \xi^o) \) and \( \Lambda^o \) is the torsion free flat classical linear connection on \( R^m \) and \( Q \) is as in Example 2. Then \( \Delta \) is smooth in \( \xi^o \) (as \( A \) is regular) and \( \Delta(\xi^o) \) is \( GL(R^m) \times \{id_G\} \)-invariant because \( A, Q, \sigma^o, \Lambda^o, 0 \) and \( e_G \) are. We prove that \( A = A^{<\Delta>} \).

It is sufficient to show that \( A(<\sigma, \Lambda>)(p) = A^{<\Delta>}(<\sigma, \Lambda>)(p) \) for any \( PB_m(G) \)-object \( P \) over \( M, \sigma \in C^\infty_M(P), \Lambda \in C^\infty_M(Q_G(LM)), p \in P, x \in M \). Because of the invariance of \( A \) and \( A^{<\Delta>} \) with respect to chart \( \varphi \) as in Example 3, we may assume that \( P = R^m \times G, M = R^m, \sigma = \sigma^o = (id_{R^m}, \xi^o), \Lambda \) is a torsion free classical linear connection on \( R^m \) with \( \Lambda(0) = \Lambda^o(0), p = (0,e_G), x = 0 \).

The invariance of \( A \) with respect to the \( PB_m(G) \)-maps \( a_t \) from the proof of Theorem 1 gives the homogeneous condition

\[
A(<\sigma^o, (a_t), \Lambda>)(0,e_G) = Ta_t \otimes T^*a_t \otimes T^*a_t(A(<\sigma^o, \Lambda>)(0,e_G)),
\]

for \( t > 0 \). Because of the non-linear Petree theorem (see Corollary 19.8 in [6]) we may assume that the Cristoffel symbols \( \Lambda \) are polynomial maps. Then by
the homogeneous function theorem (see [6]) we deduce that $A(\sigma^o, -)(0, e_G)$ depends on $\Lambda(0)$ (and similarly for $A^{<\Delta>}$ instead of $A$). So,

$$A(\sigma^o, \Lambda)(0, e_G) = A(\sigma^o, \Lambda^o)(0, e_G) = A^{<\Delta>}(\sigma^o, \Lambda^o)(0, e_G)$$

$$= A^{<\Delta>}(\sigma^o, \Lambda)(0, e_G).$$

We else describe all maps $\Delta$ from Example 3.

Let $\Delta$ be a map in question. We see that $T_{(0, e_G)}(\mathbb{R}^m \times G) = \mathbb{R}^m \oplus \text{Lie}(G)$ modulo the standard identification. Then for any $\xi \in G$, $\Delta(\xi)$ can be considered as the $GL(\mathbb{R}^m) \times \{id_{\text{Lie}(G)}\}$ invariant tensor $\Delta(\xi)$ from $(\mathbb{R}^m \oplus \text{Lie}(G)) \otimes (\mathbb{R}^m \oplus \text{Lie}(G))^* \oplus (\mathbb{R}^m \oplus \text{Lie}(G))^* = (\mathbb{R}^m \otimes \mathbb{R}^{m*} \oplus \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^m \otimes \text{Lie}(G)^* \otimes \mathbb{R}^m) \oplus (\mathbb{R}^m \otimes \text{Lie}(G)^* \oplus \mathbb{R}^m) \oplus (\mathbb{R}^m \otimes \text{Lie}(G) \otimes \mathbb{R}^m) \oplus (\text{Lie}(G) \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\text{Lie}(G) \otimes \mathbb{R}^m \otimes \text{Lie}(G)^*) \oplus (\text{Lie}(G) \otimes \mathbb{R}^m \otimes \text{Lie}(G)^*) \oplus (\text{Lie}(G) \otimes \text{Lie}(G)^* \otimes \text{Lie}(G)^*).

Thus $\Delta(\xi) = (\Delta_1(\xi), \ldots, \Delta_8(\xi))$, where $\Delta_i(\xi)$ for $i = 1, \ldots, 8$ are the respective components of $\Delta(\xi)$ with respect to the above decomposition. By the $GL(\mathbb{R}^m) \times \{id_{\text{Lie}(G)}\}$-invariance, $\Delta_2(\xi)$, $\Delta_3(\xi)$ and $\Delta_8(\xi)$ may be not zero, only. Moreover, $\Delta_5(\xi)$ may be arbitrary (smoothly depending on $\xi$), $\Delta_2(\xi) = id_{\mathbb{R}^m} \otimes \delta_2(\xi)$ and $\Delta_3(\xi) = \delta_3(\xi) \otimes id_{\mathbb{R}^m}$ (modulo the permutation), where $\delta_2(\xi)$ and $\delta_3(\xi)$ are arbitrary elements from $\text{Lie}(G)^*$ (smooth in $\xi$). Then the maps $\Delta$ from Example 3 are in bijection with the triples $(a, b, c)$ of smooth maps $a, b : G \rightarrow \text{Lie}(G)^*$ and $c : G \rightarrow \text{Lie}(G) \otimes \text{Lie}(G)^* \otimes \text{Lie}(G)^*$, $a = \delta_2$, $b = \delta_3$, $c = \Delta_8$. □

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