The natural operators of general affine connections into general affine connections

Abstract. We reduce the problem of describing all $\mathcal{M}_m$-natural operators transforming general affine connections on $m$-manifolds into general affine ones to the known description of all $GL(\mathbb{R}^m)$-invariant maps $\mathbb{R}^{m*} \otimes \mathbb{R}^m \to \otimes^k \mathbb{R}^{m*} \otimes \otimes^k \mathbb{R}^m$ for $k = 1, 3$.

Introduction. All manifolds considered in this paper are assumed to be finite dimensional, without boundaries, second countable, Hausdorff and smooth (of class $C^\infty$). Maps between manifolds are assumed to be smooth (of class $C^\infty$). The category of $m$-dimensional manifolds and their embeddings is denoted by $\mathcal{M}_m$.

A classical linear connection on a manifold $M$ is a right invariant connection $\Gamma$ on the principal fiber bundle $LM$ of linear frames of $M$. It can be considered equivalently as the corresponding $\mathbb{R}$-bilinear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that $\nabla fX Y = f \nabla X Y$ and $\nabla_X fY = X(f)Y + f \nabla_X Y$ for any map $f : M \to \mathbb{R}$ and any vector fields $X, Y \in \mathcal{X}(M)$ on $M$, see [2].

A general affine connection on $M$ is a right invariant connection $\Gamma$ on the principal fiber bundle $AM$ of affine frames of $M$. It can be equivalently considered as the corresponding pair $(\nabla, K)$ consisting of a classical linear connection $\nabla$ on $M$ and a tensor field $K$ of type $(1, 1)$ on $M$, see [2].

The general concept of natural operators can be found in [3].
In the present note, we study the problem of finding all $M_{f_m}$-natural operators $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$ transforming general affine connections $(\nabla, K)$ on $m$-manifolds $M$ into general affine connections $B(\nabla, K)$ on $M$.

Given an $M_{f_m}$-natural operator $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$, we define an $M_{f_m}$-natural operator $\Delta : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$ by

$$B(\nabla, K) = (\nabla, K) + \Delta(\nabla, K)$$

for all general affine connections $(\nabla, K)$ on $m$-manifolds $M$, and vice versa. So, to find all $M_{f_m}$-natural operators $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$ it is sufficient to find all $M_{f_m}$-natural operators $\Delta : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$ transforming general affine connections $(\nabla, K)$ on $m$-manifolds $M$ into pairs $\Delta(\nabla, K) = (\Delta^1(\nabla, K), \Delta^2(\nabla, K))$ of tensor fields $\Delta^1(\nabla, K)$ of type $(1, 2)$ and $\Delta^2(\nabla, K)$ of type $(1, 1)$ on $M$.

In the present note, we prove that the above problem of finding all $M_{f_m}$-natural operators $B : Q_{gen-af} \rightsquigarrow Q_{gen-af}$ (or $\Delta : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$) can be reduced to the one of describing all $GL(\mathbb{R}^m)$-invariant maps $\mathbb{R}^{m*} \otimes \mathbb{R}^m \to \otimes^k \mathbb{R}^{m*} \otimes \otimes^k \mathbb{R}^m$ for $k = 1, 3$.

This “reduction” is satisfactory, because the $GL(\mathbb{R}^m)$-invariant maps $\mathbb{R}^{m*} \otimes \mathbb{R}^m \to \otimes^k \mathbb{R}^{m*} \otimes \otimes^k \mathbb{R}^m$ for $k = 1, 2, 3$ are described in [1].

1. The crucial lemma. We prove the following lemma.

**Lemma 1.** There is the bijection between the set $C$ of all $M_{f_m}$-natural operators $\Delta : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$ and the set $D$ of all $GL(\mathbb{R}^m)$-invariant maps $\wedge^2 (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \to (\otimes^2 \mathbb{R}^{m*} \otimes R^m) \oplus (R^{m*} \otimes R^m)$.

**Proof.** We define a map $\Phi : C \to D$ as follows.

Any $\Delta \in C$ is determined by the values

$$\Delta(\nabla, K)(x) = (\Delta^1(\nabla, K)(x), \Delta^2(\nabla, K)(x))$$

$$\in (\otimes^2 T^*_x M \otimes T_x M) \oplus (T^*_x M \otimes T_x M)$$

for all $m$-manifolds $M$, all linear connections $\nabla$ on $M$, all tensor fields $K$ of type $(1, 1)$ on $M$ and all $x \in M$. Because of the $M_{f_m}$-invariance of $\Delta$, we may assume that $M = \mathbb{R}^m$, $x = 0$. We can even assume that $id_{\mathbb{R}^m}$ is $\nabla$-normal with center 0 (then $\nabla(0) \in \wedge^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m$ because the Christoffel symbols $\nabla^n_{jk}$ of $\nabla$ satisfy $\nabla^n_{jk}(0) + \nabla^n_{kj}(0) = 0$). Then using the invariance of $\Delta$ with respect to the homotheties $a_t = t \cdot id_{\mathbb{R}^m}$ for $t > 0$, we obtain the homogeneity condition

$$\Delta((a_t)^*, \nabla, (a_t)_*, K)(0) = (t\Delta^1(\nabla, K)(0), \Delta^2(\nabla, K)(0))$$

Because of the homogeneous function theorem [3], this type of the homogeneity implies that $\Delta(\nabla, K)(0)$ depends on $\nabla(0)$ and $j^0_0 K$ (only). Let $(\Lambda, \tau_0, \tau_1) \in (\wedge^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \equiv (\wedge^2 T^*_0 \mathbb{R}^{m*} \otimes \mathbb{R}^m)$
\( T_0 \mathbb{R}^m \oplus J^1_0 (T^* \mathbb{R}^m \otimes T \mathbb{R}^m) \), where \( \tilde{\,} \) is the usual \( GL(\mathbb{R}^m) \)-invariant identification. We put

\[
\Phi(\Delta)(\Lambda, \tau_0, \tau_1) := \Delta(\nabla, K)(0) \in (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m)
\]

(modulo the usual \( GL(\mathbb{R}^m) \)-invariant identification), where \( \nabla \) is the linear connection on \( \mathbb{R}^m \) such that the Christoffel symbols of \( \nabla \) with respect to the chart \( id_{\mathbb{R}^m} \) are constant maps and \( \nabla(0) = \nabla^m(0) + \Lambda \) and \( \nabla^m \) is the usual flat torsion free connection on \( \mathbb{R}^m \) and \( K \) is the tensor field of type \( (1,1) \) on \( \mathbb{R}^m \) such that the coefficients of \( K \) in the chart \( id_{\mathbb{R}^m} \) are polynomials of degree not more than 1 and \( j^1_0K = (\tau_0, \tau_1) \).

Since \( \Delta \) is determined by \( \Phi(\Delta) \), \( \Phi \) is injective.

It remains to show that \( \Phi \) is surjective. Let \( c : (\bigwedge^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\bigotimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \rightarrow (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \) be a \( GL(\mathbb{R}^m) \)-invariant map (an element from \( D \)). Using the usual \( GL(\mathbb{R}^m) \)-invariant identification \( \mathbb{R}^m = T_0 \mathbb{R}^m \), we have the \( GL(\mathbb{R}^m) \)-invariant map

\[
c^2 : \left( \bigwedge T^*_0 \mathbb{R}^m \otimes T_0 \mathbb{R}^m \right) \oplus (J^1_0(T^* \mathbb{R}^m \otimes T \mathbb{R}^m)) \rightarrow (\otimes^2 T^*_0 \mathbb{R}^m \otimes T_0 \mathbb{R}^m) \oplus (T^*_0 \mathbb{R}^m \otimes T_0 \mathbb{R}^m).
\]

Let \( (\nabla, K) \) be a general connection on an \( m \)-manifold \( M \). Using \( c \), we define a pair \( \Delta_c(\nabla, K) \) consisting of tensor fields \( \Delta^1_c(\nabla, K) \) of type \( (1,2) \) and \( \Delta^2_c(\nabla, K) \) of type \( (1,1) \) on \( M \) as follows. Let \( x \in M \). Consider a normal coordinate system \( \varphi \) of \( \nabla \) with center \( x \). Then \( (\varphi_*\nabla)_0 \in \bigwedge^2 T^*_0 \mathbb{R}^m \otimes T_0 \mathbb{R}^m \) modulo the obvious \( GL(\mathbb{R}^m) \)-invariant identification and \( j^1_0(\varphi_*K) \in J^1_0(T^* \mathbb{R}^m \otimes T \mathbb{R}^m) \). We put

\[
(\varphi_*\Delta_c(\nabla, K))_0 := c((\varphi_*\nabla)_0, j^1_0(\varphi_*K)).
\]

If \( \psi \) is another normal coordinate system of \( \nabla \) with center \( x \), then \( \psi = \eta \circ \varphi \) for a \( GL(\mathbb{R}^m) \)-map \( \eta \). Then \( (\psi_*\Delta_c(\nabla, K))_0 = (\varphi_*\Delta_c(\nabla, K))_0 \) because of the \( GL(\mathbb{R}^m) \)-invariance of \( c \). That is why, the definition of \( \Delta_c(\nabla, K) \) is correct. Thus we have the \( \mathcal{M}f_m \)-natural operator \( \Delta_c : Q_{gen-af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T) \). Clearly, \( \Phi(\Delta_c) = c \).

\section{The main result.}

The main result of the note is the following "reduction" theorem.

\textbf{Theorem 1.} The problem of finding all \( \mathcal{M}f_m \)-natural operators \( B : Q_{gen-af} \rightsquigarrow Q_{gen-af} \) can be reduced to the one of describing all \( GL(\mathbb{R}^m) \)-invariant maps \( \mathbb{R}^{m*} \otimes \mathbb{R}^m \rightarrow \bigotimes^k \mathbb{R}^{m*} \otimes \bigotimes^k \mathbb{R}^m \) for \( k = 1, 3 \).

\textbf{Proof.} Any \( GL(\mathbb{R}^m) \)-invariant map \( c : (\bigwedge^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\bigotimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \rightarrow (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \) is the system of \( GL(\mathbb{R}^m) \)-invariant maps

\[
c_1 : (\bigwedge^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\bigotimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \rightarrow \otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m
\]
and

\[ c_2 : (\bigwedge^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\mathbb{R}^{m*} \otimes \mathbb{R}^m) \oplus (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \rightarrow \mathbb{R}^{m*} \otimes \mathbb{R}^m. \]

Using the invariance of \( c_1 \) with respect to the homotheties \( a_t = t \text{id}_{\mathbb{R}^m} \) for \( t > 0 \), we obtain the respective homogeneity conditions. Then (by the homogeneous function theorems) \( c_1(\Lambda, \tau_0, \tau_1) \) is linear in \( \Lambda \) and \( \tau_1 \) and not necessarily linear in \( \tau_0 \). Then \( c_1 \) can be treated as the sum of \( GL(\mathbb{R}^m) \)-linear maps

\[ c_1' : \mathbb{R}^{m*} \otimes \mathbb{R}^m \rightarrow (\bigwedge^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m)^* \otimes (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \subset \otimes^3 \mathbb{R}^{m*} \otimes \otimes^3 \mathbb{R}^m \]

and

\[ c_1'' : \mathbb{R}^{m*} \otimes \mathbb{R}^m \rightarrow (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m)^* \otimes (\otimes^2 \mathbb{R}^{m*} \otimes \mathbb{R}^m) \rightarrow \otimes^3 \mathbb{R}^{m*} \otimes \otimes^3 \mathbb{R}^m. \]

By the same arguments, \( c_2(\Lambda, \tau_0, \tau_1) \) is independent of \( \Lambda \) and \( \tau_1 \). Then \( c_2 : \mathbb{R}^{m*} \otimes \mathbb{R}^m \rightarrow \mathbb{R}^{m*} \otimes \mathbb{R}^m \) is a \( GL(\mathbb{R}^m) \)-invariant map.

Now, Theorem 1 is an immediate consequence of Lemma 1. □

References