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Convolution conditions for bounded α-starlike functions of complex order

Abstract. Let A be the class of analytic functions in the unit disc U of the complex plane $\mathbb{C}$ with the normalization $f(0) = f^{'}(0) - 1 = 0$. We introduce a subclass $S_M^*(\alpha, b)$ of A, which unifies the classes of bounded starlike and convex functions of complex order. Making use of Salagean operator, a more general class $S_M^*(n, \alpha, b)$ $(n \geq 0)$ related to $S_M^*(\alpha, b)$ is also considered under the same conditions. Among other things, we find convolution conditions for a function $f \in A$ to belong to the class $S_M^*(\alpha, b)$. Several properties of the class $S_M^*(n, \alpha, b)$ are investigated.

1. Introduction. Let H denote the class of analytic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let A denote the subclass of H consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U) . \tag{1.1}$$

For functions f given by (1.1) and $g \in A$ defined by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, $z \in U$, the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad (z \in U) .$$

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Let Ω be a family of functions ω which are analytic in U and satisfy the conditions $\omega(0) = 0$, $|\omega(z)| < 1$, for every $z \in U$. Given real number M, $M > \frac{1}{2}$, let S_M^* be the class of bounded starlike functions $f \in A$ satisfying the condition

$$\left| \frac{zf'(z)}{f(z)} - M \right| \leq M \quad (z \in U).$$

This class was introduced and studied by Singh and Singh [16].

We say that $f \in A$ belongs to the class $F(b, M)$ ($b \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$, $M > \frac{1}{2}$) of bounded starlike functions of complex order, if and only if $\frac{f(z)}{z} \neq 0$ in U and

$$\left| \frac{b - 1 + \frac{zf'(z)}{f(z)}}{b} - M \right| < M \quad (z \in U).$$

The class $F(b, M)$ was introduced by Nasr and Aouf [9]. Let $C(b, M)$ ($b \in \mathbb{C}^*$, $M > \frac{1}{2}$) be the class of bounded convex functions of complex order, i.e., of functions $f \in A$ such that

$$zf'(z) \in F(b, M).$$

This class $C(b, M)$ was introduced and studied by Nasr and Aouf [8].

First let us define the class $S_M^*(\alpha, b)$ which unifies the classes of bounded starlike and convex functions of complex order.

Definition 1. We say that $f \in A$ belongs to the class $S_M^*(\alpha, b)$ ($b \in \mathbb{C}^*$, $\alpha \geq 0$, $M > \frac{1}{2}$) of bounded α-starlike functions of complex order, if and only if $\frac{f(z)f'(z)}{z} \neq 0$ in U and

$$\left| 1 + \frac{1}{b} \left(\frac{(1-\alpha)zf'(z) + \alpha z\big(zf'(z)\big)'}{(1-\alpha)f(z) + \alpha zf'(z)} - 1 \right) - M \right| < M \quad (z \in U). \tag{1.2}$$

One can easily show that $f \in S_M^*(\alpha, b)$ if and only if there is a function $g \in S_M^*$ such that

$$(1-\alpha)f(z) + \alpha zf'(z) = z\left(\frac{g(z)}{z}\right)^b \quad (z \in U). \tag{1.3}$$

It was shown in [16] that $g \in S_M^*$ if and only if for $z \in U$

$$\frac{zg'(z)}{g(z)} = \frac{1 + \omega(z)}{1 - m\omega(z)}, \quad m = 1 - \frac{1}{M}, \tag{1.4}$$

for some $\omega \in \Omega$. Thus from (1.3) and (1.4) follows that $f \in S_M^*(\alpha, b)$ if and only if

$$\frac{(1-\alpha)zf'(z) + \alpha z\big(zf'(z)\big)'}{(1-\alpha)f(z) + \alpha zf'(z)} = \frac{1 + [b(1+m) - m]\omega(z)}{1 - m\omega(z)} \quad (z \in U). \tag{1.5}$$

Taking specific values of α, b and M, we obtain the following subclasses studied by various authors:

(1) $S^*_M(0,b) \equiv F(b,M)$ and $S^*_M(1,b) \equiv C(b,M)$.

(2) $S^*_M(0,e^{-i\lambda}\cos\lambda) \equiv F_{\lambda,M}$ $(|\lambda| < \frac{\pi}{2})$ is the class of bounded λ-spirallike functions and $S^*_M(1,e^{-i\lambda}\cos\lambda) \equiv C_{\lambda,M}$ $(|\lambda| < \frac{\pi}{2})$ is the class of bounded Robertson functions that satisfy the condition $zf'(z) \in F_{\lambda,M}$, which were studied by Kulshrestha [4].

(3) $S^*_M(0,1) \equiv S^*_M$ is the class of bounded starlike functions.

(4) $S^*_\infty(0,(1-\alpha)e^{-i\lambda}\cos\lambda) \equiv S_\lambda(\alpha)$ $(|\lambda| < \frac{\pi}{2},\ 0 \le \alpha < 1)$ is the class of λ-spirallike functions of order α (see Libera [6]) and $S^*_\infty(1,(1-\alpha)e^{-i\lambda}\cos\lambda) \equiv C_\lambda(\alpha)$ $(|\lambda| < \frac{\pi}{2},\ 0 \le \alpha < 1)$ (see Kulshrestha [5] and Sizuk [15]).

(5) $S^*_\infty(0,b) \equiv S(b)$, is the class of starlike functions of complex order (see Nasr and Aouf [10]).

(6) $S^*_\infty(1,b) \equiv C(b)$ is the class of convex functions of complex order (see Wiatrowski [17] and Nasr and Aouf [7]).

(7) $S^*_\infty(0,1-\alpha) \equiv S^*(\alpha)$ $(0 \le \alpha < 1)$ is the class of starlike functions of order α and $S^*_\infty(1,1-\alpha) = C(\alpha)$ $(0 \le \alpha < 1)$ is the class of convex functions of order α (see Robertson [12]).

(8) $S^*_\infty(0,1) \equiv S^*$, $S^*_\infty(1,1) \equiv C$ and $S^*_\infty(0,e^{-i\lambda}\cos\lambda) \equiv S_\lambda(|\lambda| < \frac{\pi}{2})$ are the classes of starlike, convex and spirallike functions (More about these classes one can see in the Goodman's book [3]).

For $f \in A$, Salagean [13] introduced the following operator $D^n f$ $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,1,2,3,\ldots\})$ which is called the Salagean operator:

$$D^0 f(z) = f(z),\ D^1 f(z) = Df(z) = zf'(z),$$
$$D^n f(z) = D(D^{n-1}f(z)) \quad (z \in U)\,.$$

From the definition of $D^n f$ it follows at once that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (z \in U)\,. \tag{1.6}$$

With the aid of Salagean operator, we introduce the class $S^*_M(n,\alpha,b)$ as follows:

Definition 2. Let $M > \frac{1}{2}$, $b \in C^*$, $\alpha \ge 0$ and $n \in \mathbb{N}_0$. A function $f \in A$ is said to be in the class $S^*_M(n,\alpha,b)$ if and only if,

$$\left|1 + \frac{1}{b}\left(\frac{(1-\alpha)D^{n+1}f(z) + \alpha D^{n+2}f(z)}{(1-\alpha)D^n f(z) + \alpha D^{n+1}f(z)} - 1\right) - M\right| < M \quad (z \in U)\,.$$

We note that $S^*_M(n,0,b) \equiv H_n(b,M)$ which was studied by Aouf et al. [1].

The object of the present paper is to investigate some convolution properties of the class $S^*_M(\alpha,b)$. Using these properties, we obtain the necessary and sufficient condition for $f \in A$ to belong to the class $S^*_M(n,\alpha,b)$. Also we establish the relationship among the classes $S_M^*(n+1,\alpha,b)$ and $S_M^*(n,\alpha,b)$. These results generalize the related works of some authors.

2. Convolution conditions. Unless otherwise mentioned, we assume throughout this article that $b\in\mathbb{C}^*$, $M>\frac{1}{2}$, $\alpha\geq 0$ and $n\in\mathbb{N}_0$.

Theorem 1. *A function f of the form* (1.1) *is in the class $S_M^*(\alpha,b)$ if and only if*

$$\frac{1}{z}\left[f(z)*\left\{(1-\alpha)\frac{z-Cz^2}{(1-z)^2}+\alpha\frac{z+(1-2C)z^2}{(1-z)^3}\right\}\right]\neq 0 \quad (z\in U) \tag{2.1}$$

where $C=C_\theta=\frac{e^{-i\theta}+[b(1+m)-m]}{b(1+m)}$, $\theta\in[0,2\pi)$.

Proof. A function f is in the class $S_M^*(\alpha,b)$ if and only if

$$\frac{(1-\alpha)zf^{'}(z)+\alpha z\big(zf^{'}(z)\big)^{'}}{(1-\alpha)f(z)+\alpha zf^{'}(z)}=\frac{1+[b(1+m)-m]\omega(z)}{1-m\omega(z)} \quad (z\in U),$$

where $m=1-\frac{1}{M}$, which is equivalent to

$$\frac{z\Big[(1-\alpha)f(z)+\alpha zf^{'}(z)\Big]^{'}}{(1-\alpha)f(z)+\alpha zf^{'}(z)}\neq\frac{1+[b(1+m)-m]e^{i\theta}}{1-me^{i\theta}} \tag{2.2}$$

($z\in U$, $\theta\in[0,2\pi)$) and further to

$$\begin{aligned} &z\Big[(1-\alpha)f(z)+\alpha zf^{'}(z)\Big]^{'}\Big(1-me^{i\theta}\Big)\\ &\quad-\Big[(1-\alpha)f(z)+\alpha zf^{'}(z)\Big]\Big(1+[b(1+m)-m]e^{i\theta}\Big)\neq 0 \end{aligned} \tag{2.3}$$

for some $z\in U$ and $\theta\in[0,2\pi)$. It is well known that

$$f(z)=f(z)*\frac{z}{(1-z)},\quad zf'(z)=f(z)*\frac{z}{(1-z)^2} \quad (z\in U). \tag{2.4}$$

Using (2.4), it is easy to verify that

$$(1-\alpha)f(z)+\alpha zf^{'}(z)=f(z)*\frac{z-(1-\alpha)z^2}{(1-z)^2} \quad (z\in U). \tag{2.5}$$

Since $z(f*g)^{'}=f*zg^{'}$, we have

$$z\Big[(1-\alpha)f(z)+\alpha zf^{'}(z)\Big]^{'}=f(z)*\frac{z+(2\alpha-1)z^2}{(1-z)^3} \quad (z\in U). \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.3), we get

$$\begin{aligned} &\frac{1}{z}[f(z)*\{-(1-\alpha)(1-z)[b(1+m)e^{i\theta}z\\ &\qquad-\Big(1+[b(1+m)-m]e^{i\theta}\Big)z^2]\\ &\qquad-\alpha(1-z)b(1+m)e^{i\theta}z+2\alpha(1-me^{i\theta})z^2\}/(1-z)^3]\neq 0 \end{aligned} \tag{2.7}$$

$(z \in U,\ \theta \in [0, 2\pi))$ i.e., equivalently,

$$\frac{1}{z}[f(z) * \{-(1-\alpha)(1-z)[b(1+m)e^{i\theta}z - \left(1 + [b(1+m) - m]e^{i\theta}\right) z^2]$$
$$- \alpha[b(1+m)e^{i\theta}z + \{b(1+m)e^{i\theta}$$
$$- 2\left(1 + [b(1+m) - m]e^{i\theta}\right)\}z^2]\}/(1-z)^3] \neq 0$$

for some $z \in U$ and $\theta \in [0, 2\pi)$. Thus (2.7) can be rewritten as follows

$$\frac{1}{z}\left[f(z) * \left\{(1-\alpha)\frac{z - \frac{e^{-i\theta}+[b(1+m)-m]}{b(1+m)}z^2}{(1-z)^2}\right.\right.$$
$$\left.\left. + \alpha\frac{z + \left(1 - 2\frac{e^{-i\theta}+[b(1+m)-m]}{b(1+m)}\right)z^2}{(1-z)^3}\right\}\right] \neq 0$$

where $z \in U$, $\theta \in [0, 2\pi)$. Hence the proof of Theorem 1 is complete. □

Remark 1.

(1) Taking $\alpha = 0$ in Theorem 1, we obtain the result obtained by El-Ashwah [2, Theorem 2.1].
(2) Taking $\alpha = 1$ in Theorem 1, we obtain the result obtained by El-Ashwah [2, Theorem 2.4].
(3) Taking $\alpha = 1$, $b = 1 - \beta$ $(0 \leq \beta < 1)$, $M = \infty$ and $e^{i\theta} = x$ in Theorem 1, we obtain the result obtained by Silverman et al. [14, Theorem 1].
(4) Taking $\alpha = 0$, $b = 1 - \beta$ $(0 \leq \beta < 1)$, $M = \infty$ and $e^{i\theta} = x$ in Theorem 1, we obtain the result obtained by Silverman et al. [14, Theorem 2].
(5) Taking $\alpha = 1$, $b = e^{-i\lambda}\cos\lambda$ $(|\lambda| < 1)$, $M = \infty$ and $e^{i\theta} = x$ in Theorem 1, we obtain the result obtained by Padmanabhan and Ganesan [11, Theorem 1] with $B = -1$ and $A = 1$.
(6) Taking $\alpha = 0$, $b = e^{-i\lambda}\cos\lambda$ $(|\lambda| < 1)$, $M = \infty$ and $e^{i\theta} = x$ in Theorem 1, we obtain the result obtained by Padmanabhan and Ganesan [11, Theorem 2] with $B = -1$ and $A = 1$.

Theorem 2. *A function f of the form* (1.1) *is in the class $S^*_M(n, \alpha, b)$ if and only if*

$$1 - \sum_{k=2}^{\infty} k^n \frac{(k-1)[e^{-i\theta} - m] - b(1+m)}{b(1+m)}[(1-\alpha) + \alpha k]a_k z^{k-1} \neq 0 \tag{2.8}$$

for all $\theta \in [0, 2\pi)$ and $z \in U$.

Proof. Note that $f \in S_M^*(n, \alpha, b)$ if and only if $D^n f \in S_M^*(\alpha, b)$. Thus from Theorem 1, we have $f \in S_M^*(n, \alpha, b)$ if and only if

$$\frac{1}{z}\left[D^n f(z) * \left\{(1-\alpha)\frac{z - Cz^2}{(1-z)^2} + \alpha\frac{z + (1-2C)z^2}{(1-z)^3}\right\}\right] \neq 0 \quad (z \in U) \tag{2.9}$$

where $C = C_\theta = \frac{e^{-i\theta} + [b(1+m) - m]}{b(1+m)}$ and $\theta \in [0, 2\pi)$, i.e., if and only if

$$\begin{aligned} &\frac{1}{z}\left[D^n f(z) * \left\{(1-\alpha)\left[\frac{Cz}{1-z} + \frac{(1-C)z}{(1-z)^2}\right]\right.\right. \\ &\qquad \left.\left. + \alpha\left[\frac{2(1-C)z}{(1-z)^3} - \frac{(1-2C)z}{(1-z)^2}\right]\right\}\right] \neq 0 \end{aligned} \tag{2.10}$$

$(z \in U)$. Since for $z \in U$,

$$\frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k, \quad \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} kz^k$$

and

$$\frac{z}{(1-z)^3} = z + \sum_{k=2}^{\infty} \frac{k(k+1)}{2} z^k,$$

from (1.6) and (2.10) it follows that

$$1 - \sum_{k=2}^{\infty} k^n \frac{(k-1)[e^{-i\theta} - m] - b(1+m)}{b(1+m)}[(1-\alpha) + \alpha k]a_k z^{k-1} \neq 0$$

$(z \in U)$. This completes the proof of Theorem 2. □

Theorem 3. *If $f \in A$ satisfies the inequality*

$$\sum_{k=2}^{\infty}(k - 1 + |b|)[(1-\alpha) + \alpha k]k^n |a_k| \leq |b|, \tag{2.11}$$

then $f \in S_M^*(n, \alpha, b)$.

Proof. Since

$$\left|\frac{(k-1)[e^{-i\theta} - m] - b(1+m)}{b(1+m)}\right| \leq \frac{(k-1+|b|)}{|b|},$$

so

$$\begin{aligned} &\left|1 - \sum_{k=2}^{\infty} \frac{(k-1)[e^{-i\theta} - m] - b(1+m)}{b(1+m)}[(1-\alpha) + \alpha k]k^n a_k z^{k-1}\right| \\ &\geq 1 - \sum_{k=2}^{\infty} \left|\frac{(k-1)[e^{-i\theta} - m] - b(1+m)}{b(1+m)}\right| [(1-\alpha) + \alpha k]k^n |a_k| |z|^{k-1} \\ &\geq 1 - \sum_{k=2}^{\infty} \frac{k-1+|b|}{|b|}[(1-\alpha) + \alpha k]k^n |a_k| > 0 \end{aligned}$$

$(z \in U)$. Thus (2.8) holds, which ends the proof. □

Theorem 4. $S^*_M(n+1,\alpha,b) \subset S^*_M(n,\alpha,b)$.

Proof. Let $f \in S^*_M(n+1,\alpha,b)$. By Theorem 2, we have

$$(2.12)\quad 1-\sum_{k=2}^{\infty}\frac{(k-1)[e^{-i\theta}-m]-b(1+m)}{b(1+m)}[(1-\alpha)+\alpha k]k^{n+1}a_k z^{k-1} \neq 0$$

$(z \in U)$, which is equivalent to

$$(2.13)\quad \left[1+\sum_{k=2}^{\infty}kz^{k-1}\right] * \left[1-\sum_{k=2}^{\infty}k^n\frac{(k-1)[e^{-i\theta}-m]-b(1+m)}{b(1+m)}[(1-\alpha)+\alpha k]a_k z^{k-1}\right] \neq 0$$

$(z \in U)$. Since

$$\left[1+\sum_{k=2}^{\infty}kz^{k-1}\right] * \left[1+\sum_{k=2}^{\infty}\frac{1}{k}z^{k-1}\right] = 1+\sum_{k=2}^{\infty}z^{k-1} \quad (z \in U),$$

by using the property, if $f \neq 0$ and $g * h \neq 0$, then $f * (g * h) \neq 0$, (2.13) can be written as

$$(2.14)\quad 1-\sum_{k=2}^{\infty}k^n\frac{(k-1)[e^{-i\theta}-m]-b(1+m)}{b(1+m)}[(1-\alpha)+\alpha k]a_k z^{k-1} \neq 0$$

$(z \in U)$. Thus the assertion follows from Theorem 2. □

(1) Putting $\alpha = 0$ in Theorems 2, 3 and 4, we get the results obtained by El-Ashwah [2, Theorems 2.7, 3.1 and 3.4].
(2) Putting $\alpha = 1$ in Theorems 2, 3 and 4, we get the results obtained by El-Ashwah [2, Theorems 2.8, 3.2 and 3.5].

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