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On branchwise commutative pseudo-BCH algebras

Abstract. Basic properties of branches of pseudo-BCH algebras are described. Next, the concept of a branchwise commutative pseudo-BCH algebra is introduced. Some conditions equivalent to branchwise commutativity are given. It is proved that every branchwise commutative pseudo-BCH algebra is a pseudo-BCI algebra.

1. Introduction. In 1966, Imai and Iséki ([9, 13]) introduced BCK and BCI algebras. In 1983, Hu and Li ([8]) defined BCH algebras. It is known that BCK and BCI algebras are contained in the class of BCH algebras. In [11, 12], Iorgulescu introduced many interesting generalizations of BCI and BCK algebras (see also [10]).

In 2001, Georgescu and Iorgulescu ([7]) defined pseudo-BCK algebras as an extension of BCK algebras. In 2008, Dudek and Jun ([1]) introduced pseudo-BCI algebras as a natural generalization of BCI algebras and of pseudo-BCK algebras. These algebras have also connections with other algebras of logic such as pseudo-MV algebras and pseudo-BL algebras defined by Georgescu and Iorgulescu in [5] and [6], respectively. Recently, Walendziak ([14]) introduced pseudo-BCH algebras as an extension of BCH algebras. In [15, 16], he studied ideals in such algebras.

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In this paper we consider branches of pseudo-BCH algebras and introduce the concept of a branchwise commutative pseudo-BCH algebra. We show that every such algebra is a pseudo-BCI algebra. We also give some conditions equivalent to branchwise commutativity. Finally, we obtain a system of identities defining the class of branchwise commutative pseudo-BCH algebras.

2. Preliminaries. We recall that an algebra $\mathfrak{X} = (X; *, 0)$ of type $(2, 0)$ is called a *BCH algebra* if it satisfies the following axioms:

(BCH-1) $x * x = 0$;
(BCH-2) $(x * y) * z = (x * z) * y$;
(BCH-3) $x * y = y * x = 0 \Longrightarrow x = y$.

A BCH algebra $\mathfrak{X}$ is said to be a *BCI algebra* if it satisfies the identity

(BCI) $((x * y) * (x * z)) * (z * y) = 0$.

A *BCK algebra* is a BCI algebra $\mathfrak{X}$ satisfying the law $0 * x = 0$.

Definition 2.1 ([1]). A *pseudo-BCI algebra* is a structure $\mathfrak{X} = (X; \leqslant, *, \diamond, 0)$, where "$\leqslant$" is a binary relation on the set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, satisfying the axioms:

(pBCI-1) $(x * y) \diamond (x * z) \leqslant z * y, \quad (x \diamond y) * (x \diamond z) \leqslant z \diamond y$;
(pBCI-2) $x * (x \diamond y) \leqslant y, \quad x \diamond (x * y) \leqslant y$;
(pBCI-3) $x \leqslant x$;
(pBCI-4) $x \leqslant y,\ y \leqslant x \Longrightarrow x = y$;
(pBCI-5) $x \leqslant y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

A pseudo-BCI algebra $\mathfrak{X}$ is called a *pseudo-BCK algebra* if it satisfies the identities

(pBCK) $0 * x = 0 \diamond x = 0$.

Definition 2.2 ([14]). A *pseudo-BCH algebra* is an algebra $\mathfrak{X} = (X; *, \diamond, 0)$ of type $(2, 2, 0)$ satisfying the axioms:

(pBCH-1) $x * x = x \diamond x = 0$;
(pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;
(pBCH-3) $x * y = y \diamond x = 0 \Longrightarrow x = y$;
(pBCH-4) $x * y = 0 \Longleftrightarrow x \diamond y = 0$.

We define a binary relation $\leqslant$ on X by

$$x \leqslant y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0.$$

Throughout this paper $\mathfrak{X}$ will denote a pseudo-BCH algebra.

Example 2.3 ([14], Example 4.12). Let $X = \{0, a, b, c, d\}$. Define binary operations $*$ and $\diamond$ on X by the following tables:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	d
b	b	b	0	0	d
c	c	b	c	0	d
d	d	d	d	d	0

$\diamond$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	d
b	b	b	0	0	d
c	c	c	a	0	d
d	d	d	d	d	0

Then $\mathfrak{X} = (X; *, \diamond, 0)$ is a pseudo-BCH algebra.

Let $\mathfrak{X} = (X; *, \diamond, 0)$ be a pseudo-BCH algebra satisfying (pBCK), and let $(G; \cdot, 1)$ be a group. Denote $Y = G - \{1\}$ and suppose that $X \cap Y = \emptyset$. Define the binary operations $*$ and $\diamond$ on $X \cup Y$ by

$$x * y = \begin{cases} x * y & \text{if } x, y \in X \\ xy^{-1} & \text{if } x, y \in Y \text{ and } x \neq y \\ 0 & \text{if } x, y \in Y \text{ and } x = y \\ y^{-1} & \text{if } x \in X,\ y \in Y \\ x & \text{if } x \in Y,\ y \in X \end{cases} \tag{1}$$

and

$$x \diamond y = \begin{cases} x \diamond y & \text{if } x, y \in X \\ y^{-1}x & \text{if } x, y \in Y \text{ and } x \neq y \\ 0 & \text{if } x, y \in Y \text{ and } x = y \\ y^{-1} & \text{if } x \in X,\ y \in Y \\ x & \text{if } x \in Y,\ y \in X. \end{cases} \tag{2}$$

Then $(X \cup Y; *, \diamond, 0)$ is a pseudo-BCH algebra (see [15]).

Example 2.4. Consider the set $X = \{0, a, b, c\}$ with the operation $*$ defined by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	a
c	c	a	a	0

By simple calculation we can get that $\mathfrak{X} = (X; *, 0)$ is a BCH algebra. Let $\mathfrak{G}$ be the group of all permutations of $\{1, 2, 3\}$. We have $G = \{\imath, d, e, f, g, h\}$, where $\imath = (1)$, $d = (12)$, $e = (13)$, $f = (23)$, $g = (123)$, and $h = (132)$. Applying (1) and (2), we obtain the following tables:

$*$	0	a	b	c	d	e	f	g	h
0	0	0	0	0	d	e	f	h	g
a	a	0	0	0	d	e	f	h	g
b	b	a	0	a	d	e	f	h	g
c	c	a	a	0	d	e	f	h	g
d	d	d	d	d	0	h	g	e	f
e	e	e	e	e	g	0	h	f	d
f	f	f	f	f	h	g	0	d	e
g	g	g	g	g	e	f	d	0	h
h	h	h	h	h	f	d	e	g	0

and

$\diamond$	0	a	b	c	d	e	f	g	h
0	0	0	0	0	d	e	f	h	g
a	a	0	0	0	d	e	f	h	g
b	b	a	0	a	d	e	f	h	g
c	c	a	a	0	d	e	f	h	g
d	d	d	d	d	0	g	h	f	e
e	e	e	e	e	h	0	g	d	f
f	f	f	f	f	g	h	0	e	d
g	g	g	g	g	f	d	e	0	h
h	h	h	h	h	e	f	d	g	0

Then $(\{0, a, b, c, d, e, f, g, h\}; *, \diamond, 0)$ is a pseudo-BCH algebra.

From [14] it follows that in any pseudo-BCH algebra $\mathfrak{X}$, for all $x, y \in X$, we have:

(P1) $x \leqslant x$,
(P2) $x \leqslant y,\ y \leqslant x \Longrightarrow x = y$,
(P3) $x * (x \diamond y) \leqslant y$ and $x \diamond (x * y) \leqslant y$,
(P4) $x \leqslant 0 \Longrightarrow x = 0$,
(P5) $x * 0 = x \diamond 0 = x$,
(P6) $0 * x = 0 \diamond x$,
(P7) $x \leqslant y \Longrightarrow 0 * x = 0 \diamond y$,
(P8) $0 * (x * y) = (0 * x) \diamond (0 * y)$,
(P9) $0 * (x \diamond y) = (0 * x) * (0 * y)$.

Remark. By Theorem 3.4 of [14], a pseudo-BCH algebra is a pseudo-BCI algebra if and only if it satisfies the following implication:

(*) $$x \leqslant y \Longrightarrow (x * z \leqslant y * z, x \diamond z \leqslant y \diamond z).$$

Proposition 2.5. *For a pseudo-BCH algebra $\mathfrak{X}$ the following conditions are equivalent:*

(a) *$\mathfrak{X}$ is a pseudo-BCI algebra,*
(b) *$\mathfrak{X}$ satisfies axiom* (pBCI-1)*,*
(c) *$\mathfrak{X}$ satisfies condition* (*)*.*

Proof. The equivalence of (a) and (c) follows from the above remark.

(a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (a): By assumption, $\mathfrak{X}$ satisfies (pBCI-1) and (pBCI-5). The axioms (pBCI-2)–(pBCI-4) follow from the properties (P1)–(P3). $\square$

3. Atoms and branches. An element a of $\mathfrak{X}$ is called an *atom* if $x \leqslant a$ implies $x = a$ for all $x \in X$, that is, a is a minimal element of $(X; \leqslant)$. Let us denote by $\mathrm{A}(\mathfrak{X})$ the set of all atoms of $\mathfrak{X}$. By (P4), $0 \in \mathrm{A}(\mathfrak{X})$.

Proposition 3.1 ([14], Propositions 4.1 and 4.2). *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $a \in X$. Then the following conditions are equivalent:*

(i) *a is an atom,*
(ii) $x \diamond (x * a) = a$ *for all $x \in X$,*
(iii) $0 \diamond (0 * a) = a$,
(iv) $x * (x \diamond a) = a$ *for all $x \in X$,*
(v) $0 * (0 \diamond a) = a$.

Proposition 3.2 ([14], Proposition 4.3). *Let $\mathfrak{X}$ be a pseudo-BCH algebra and let $a \in X$. Then a is an atom if and only if there is an element $x \in X$ such that $a = 0 * x$.*

As a consequence of Proposition 3.2, we obtain

Corollary 3.3. *For every $x \in X$, we have $0 * x \in \mathrm{A}(\mathfrak{X})$.*

For $x \in X$, set

$$\overline{x} = 0 \diamond (0 * x).$$

By (P6), $\overline{x} = 0 * (0 * x) = 0 \diamond (0 \diamond x) = 0 * (0 \diamond x)$. Note that the map $\varphi(x) = 0*(0*x)$ was introduced in [17] for BZ algebras (such algebras are a generalization of BCI algebras). Different properties of this map were used in many papers (for example, [18], [2] and [3]).

Proposition 3.4 ([14], Proposition 4.4). *Let $\mathfrak{X}$ be a pseudo-BCH algebra. For any $x, y \in X$ we have:*

(i) $\overline{x * y} = \overline{x} * \overline{y}$,
(ii) $\overline{x \diamond y} = \overline{x} \diamond \overline{y}$,
(iii) $\overline{\overline{x}} = \overline{x}$.

For BZ algebras, (iii) was proved in [17]. In [14], the set $\{x \in X : x = \overline{x}\}$ is called the *centre* of $\mathfrak{X}$ and it is denoted by $\mathrm{Cen}\mathfrak{X}$. We conclude from Proposition 3.1 that $\mathrm{Cen}\mathfrak{X} = \mathrm{A}(\mathfrak{X})$. Then $\mathrm{A}(\mathfrak{X}) = \{\overline{x} : x \in X\}$. By Proposition 3.4, $\mathrm{A}(\mathfrak{X})$ is a subalgebra of $\mathfrak{X}$.

For any pseudo-BCH algebra $\mathfrak{X}$, we set

$$\mathrm{K}(\mathfrak{X}) = \{x \in X : 0 \leqslant x\}.$$

From Corollary 4.19 of [14] it follows that $\mathrm{K}(\mathfrak{X})$ is a subalgebra of $\mathfrak{X}$.

Observe that

$$\mathrm{A}(\mathfrak{X}) \cap \mathrm{K}(\mathfrak{X}) = \{0\}.$$

Indeed, $0 \in \mathrm{A}(\mathfrak{X}) \cap \mathrm{K}(\mathfrak{X})$ and if $x \in \mathrm{A}(\mathfrak{X}) \cap \mathrm{K}(\mathfrak{X})$, then $x = 0 * (0 * x) = 0 * 0 = 0$.

Lemma 3.5. *Let $x, y \in X$. If $x * y \in \mathrm{K}(\mathfrak{X})$, then $y * x$, $x \diamond y$, $y \diamond x \in \mathrm{K}(\mathfrak{X})$.*

Proof. Let $x * y \in \mathrm{K}(\mathfrak{X})$. Then $0 * (x * y) = 0$. We deduce from (P8) that $(0 * x) \diamond (0 * y) = 0$, and hence $0 * x \leqslant 0 * y$. Since $0 * x, 0 * y \in \mathrm{A}(\mathfrak{X})$ (see Corollary 3.3), we have $0 * x = 0 * y$. Consequently,

$$0 * (y * x) = (0 * y) \diamond (0 * x) = (0 * y) \diamond (0 * y) = 0,$$

that is, $0 * (y * x) = 0$. Applying (P9), we also deduce that $0 * (x \diamond y) = 0$ and $0 * (y \diamond x) = 0$. Therefore, $y * x$, $x \diamond y$, $y \diamond x \in \mathrm{K}(\mathfrak{X})$. □

For any element a of a pseudo-BCH-algebra $\mathfrak{X}$, we define a subset $\mathrm{V}(a)$ of X as

$$\mathrm{V}(a) = \{x \in X : a \leq x\}.$$

Note that $\mathrm{V}(a) \neq \emptyset$, because $a \leq a$ gives $a \in \mathrm{V}(a)$. Furthermore, $\mathrm{V}(0) = \mathrm{K}(\mathfrak{X})$. If $a \in \mathrm{A}(\mathfrak{X})$, then the set $\mathrm{V}(a)$ is called a *branch* of $\mathfrak{X}$ determined by element a.

Example 3.6. Let $\mathfrak{X} = (\{0, a, b, c, d\}; *, \diamond, 0)$ be the pseudo-BCH algebra given in Example 2.3. It is easily seen that $\mathrm{A}(\mathfrak{X}) = \{0, d\}$ and $\mathfrak{X}$ has two branches $\mathrm{V}(0) = \{0, a, b, c\}$ and $\mathrm{V}(d) = \{d\}$.

Example 3.7. Let $\mathfrak{X} = (\{0, a, b, c, d, e, f, g, h\}; *, \diamond, 0)$ be the pseudo-BCH algebra from Example 2.4. Obviously, $\mathrm{A}(\mathfrak{X}) = \{0, d, e, f, g, h\}$. The algebra $\mathfrak{X}$ has the following branches: $\mathrm{V}(0) = \{0, a, b, c\}$, $\mathrm{V}(d) = \{d\}$, $\mathrm{V}(e) = \{e\}$, $\mathrm{V}(f) = \{f\}$, $\mathrm{V}(g) = \{g\}$, $\mathrm{V}(h) = \{h\}$.

Proposition 3.8 ([14], Proposition 4.23). *Let $\mathfrak{X}$ be a pseudo-BCH algebra. Then:*

(i) $X = \bigcup\{\mathrm{V}(a) : a \in \mathrm{A}(\mathfrak{X})\}$.

(ii) *if $a, b \in \mathrm{A}(\mathfrak{X})$ and $a \neq b$, then $V(a) \cap \mathrm{V}(b) = \emptyset$.*

Proposition 3.9. *Two elements x, y are in the same branch of $\mathfrak{X}$ if and only if $x * y \in \mathrm{K}(\mathfrak{X})$ (or equivalently, $x \diamond y \in \mathrm{K}(\mathfrak{X})$).*

Proof. If x and y are in the same branch $\mathrm{V}(a)$, then $a \leqslant x$ and $a \leqslant y$. By (P6) and (P7), $0 * x = 0 * a = 0 * y$. Applying (P8), we obtain $0 * (x * y) = (0 * x) \diamond (0 * y) = 0$. Thus $0 \leqslant x * y$, that is, $x * y \in \mathrm{K}(\mathfrak{X})$.

Conversely, suppose that $x * y \in \mathrm{K}(\mathfrak{X})$ and $x \in \mathrm{V}(a)$, $y \in \mathrm{V}(b)$ for some $a, b \in \mathrm{A}(\mathfrak{X})$. Hence $a \leqslant x$ and $b \leqslant y$. Using (P6) and (P7), we get $0*a = 0*x$ and $0 * b = 0 * y$. Therefore, $a = \overline{x}$ and $b = \overline{y}$. From Proposition 3.4 we have $\overline{x * y} = \overline{x} * \overline{y} = a * b$ and $\overline{y \diamond x} = b \diamond a$. Since $x * y \in \mathrm{K}(\mathfrak{X})$ and also $y \diamond x \in \mathrm{K}(\mathfrak{X})$ (see Lemma 3.5) we conclude that $\overline{x * y} = \overline{y \diamond x} = 0$. Therefore, $a * b = b \diamond a = 0$ which gives $a = b$. So x and y are in the same branch. □

Proposition 3.10. *Comparable elements of $\mathfrak{X}$ are in the same branch.*

Proof. Let $x, y \in X$ and let $x \leqslant y$. Then $x * y = 0 \in \mathrm{K}(\mathfrak{X})$. By Proposition 3.9, x and y are in the same branch. $\square$

Proposition 3.11. *If elements x and y are comparable, then $x * y$, $y * x$, $x \diamond y$, $y \diamond x \in \mathrm{K}(\mathfrak{X})$.*

Proof. From Propositions 3.10 and 3.9 we see that $x * y \in \mathrm{K}(\mathfrak{X})$ and hence $y * x$, $x \diamond y$, $y \diamond x \in \mathrm{K}(\mathfrak{X})$ by Lemma 3.5. $\square$

4. Branchwise commutativity. A pseudo-BCH algebra $\mathfrak{X}$ is said to be *commutative* if for all $x, y \in X$, it satisfies the following identities:

$$x * (x \diamond y) = y * (y \diamond x), \tag{3}$$

$$x \diamond (x * y) = y \diamond (y * x). \tag{4}$$

Proposition 4.1. *Every commutative pseudo-BCH algebra is a pseudo-BCK algebra.*

Proof. Let $\mathfrak{X}$ be a commutative pseudo-BCH algebra. First observe that $\mathfrak{X}$ satisfies (pBCK). Let $x \in X$. Applying (pBCH-1), (P5) and (P3), we obtain

$$0 = x * x = x * (x \diamond 0) = 0 * (0 \diamond x) \leqslant x.$$

Then $0 * x = 0 \diamond x = 0$, that is, (pBCK) holds.

Now we show that $\mathfrak{X}$ satisfies (pBCI-1). Let $x, y \in X$. We have

$$\begin{aligned}
((x * y) \diamond (x * z)) * (z * y) &= ((x \diamond (x * z)) * y) * (z * y) && \text{[by (pBCH-2)]}\\
&= ((z \diamond (z * x)) * y) * (z * y) && \text{[by (4)]}\\
&= ((z * y) * (z * y)) \diamond (z * x) && \text{[by (pBCH-2)]}\\
&= 0 \diamond (z * x) && \text{[by (pBCH-1)]}\\
&= 0 && \text{[by (pBCK)]}
\end{aligned}$$

and hence $(x * y) \diamond (x * z) \leqslant (z * y)$. Similarly, $(x \diamond y) * (x \diamond z) \leqslant z \diamond y$. Thus (pBCI-1) holds in $\mathfrak{X}$. We conclude from Proposition 2.5 that $\mathfrak{X}$ is a pseudo-BCI algebra, and finally that it is a pseudo-BCK algebra. $\square$

Corollary 4.2. *Commutative pseudo-BCH algebras coincide with commutative pseudo-BCK algebras.*

In [4], G. Dymek introduced the notion of branchwise commutative pseudo-BCI algebras. Following [4], we say that a pseudo-BCH algebra $\mathfrak{X}$ is *branchwise commutative* if identities (3) and (4) hold for x and y belonging to the same branch. Clearly, any commutative pseudo-BCH algebra is branchwise commutative.

Remark. Note that the pseudo-BCH algebra from Example 2.4 is branchwise commutative but it is not commutative, since $d \diamond (d * a) = 0 \neq d = a \diamond (a * d)$.

The algebra given in Example 2.3 is not branchwise commutative. Indeed, $a * (a \diamond c) = a$ but $c * (c \diamond a) = 0$.

Proposition 4.3 ([4], Theorem 3.2). *A pseudo-BCI algebra $(X; \leqslant, *, \diamond, 0)$ is branchwise commutative if and only if for all $x, y \in X$, satisfies the following condition:*

(BC) $$x \leqslant y \Longrightarrow x = y \diamond (y * x) = y * (y \diamond x).$$

Lemma 4.4. *If $\mathfrak{X}$ satisfies* (BC)*, then $\mathfrak{X}$ is a pseudo-BCI algebra.*

Proof. Let $x, y \in X$ and $x \leqslant y$. We have

$$\begin{aligned}(x * z) \diamond (y * z) &= ((y \diamond (y * x)) * z) * (y * z) && [\text{since } x = y \diamond (y * x)]\\ &= ((y * z) \diamond (y * x)) * (y * z) && [\text{by (pBCH-2)}]\\ &= ((y * z) * (y * z)) \diamond (y * x) && [\text{by (pBCH-2)}]\\ &= 0 \diamond (y * x) && [\text{by (pBCH-1)}].\end{aligned}$$

Since elements x and y are comparable, by Proposition 3.11, $y * x \in \mathrm{K}(\mathfrak{X})$. Therefore, $0 \diamond (y * x) = 0$, and hence $(x * z) \diamond (y * z) = 0$. Consequently, $x * z \leqslant y * z$. Similarly, $x \diamond z \leqslant y \diamond z$. From Proposition 2.5 it follows that $\mathfrak{X}$ is a pseudo-BCI algebra. □

As a consequence of the above lemma and Proposition 4.3, we obtain:

Proposition 4.5. *If a pseudo-BCH algebra satisfies* (BC)*, then it is branchwise commutative.*

Theorem 4.6. *Any branchwise commutative pseudo-BCH algebra is a pseudo-BCI algebra.*

Proof. Let $\mathfrak{X}$ be a brachwise commutative pseudo-BCH algebra. Let $x, y \in X$ and $x \leqslant y$. Then $x * y = 0$. By Proposition 3.10, elements x and y are in the same branch. Since $\mathfrak{X}$ is brachwise commutative, we obtain

$$y \diamond (y * x) = x \diamond (x * y) = x \diamond 0 = x.$$

Similarly, we prove that $x = y * (y \diamond x)$. Thus condition (BC) holds in $\mathfrak{X}$. From Lemma 4.4 we conclude that $\mathfrak{X}$ is a pseudo-BCI algebra. □

Corollary 4.7. *Branchwise commutative pseudo-BCH algebras coincide with branchwise commutative pseudo-BCI algebras.*

As a consequence of Corollary 4.7, all results holding for branchwise commutative pseudo-BCI algebras also hold for brachwise commutative pseudo-BCH algebras.We recall some of these results:

Proposition 4.8 ([4]). *Let $\mathfrak{X}$ be a branchwise commutative pseudo-BCH/ BCI algebra. Then:*

(i) *for all $x, y \in X$, we have*

$$x \diamond (x * y) = y \diamond (y * (x \diamond (x * y))), \tag{5}$$

$$x * (x \diamond y) = y * (y \diamond (x * (x \diamond y))). \tag{6}$$

(ii) *for all x and y belonging to the same branch,*

$$x * y = x * (y \diamond (y * x)), \tag{7}$$

$$x \diamond y = x \diamond (y * (y \diamond x)). \tag{8}$$

(iii) *each branch of $\mathfrak{X}$ is a semilattice with respect to the operation $\wedge$ defined by $x \wedge y = y \diamond (y * x) = y * (y \diamond x)$.*

Theorem 4.9. *Let $\mathfrak{X}$ be a pseudo-BCH algebra. The following are equivalent:*

(a) *$\mathfrak{X}$ is branchwise commutative,*
(b) *$\mathfrak{X}$ satisfies* (BC),
(c) *$\mathfrak{X}$ satisfies* (5) *and* (6),
(d) *the identities* (7) *and* (8) *hold for all x and y belonging to the same branch of $\mathfrak{X}$,*
(e) *each branch of $\mathfrak{X}$ is a semilattice with respect to the operation $\wedge$ defined by $x \wedge y = y \diamond (y * x) = y * (y \diamond x)$.*

Proof. Let $\mathfrak{X}$ be a branchwise commutative pseudo-BCH algebra. Then, by Theorem 4.6, $\mathfrak{X}$ is a branchwise commutative pseudo-BCI algebra. From Propositions 4.3 and 4.8 we deduce that (a) implies (b), (c), (d) and (e).

(c) $\Longrightarrow$ (b): Let $x, y \in X$ and $x \leqslant y$. Then $x * y = 0$. From (5) we see that $x = y \diamond (y * x)$. Similarly, from (6) we get $x = y * (y \diamond x)$. Therefore, (BC) holds in $\mathfrak{X}$.

(d) $\Longrightarrow$ (b): Suppose that $x \leqslant y$. By Proposition 3.10, elements x and y are in the same branch. Putting $x * y = 0$ in (7) and $x \diamond y = 0$ in (8), we get $0 = x * (y \diamond (y * x)) = x \diamond (y * (y \diamond x))$. Hence $x \leqslant y \diamond (y * x)$ and $x \leqslant y * (y \diamond x)$. Applying (P3), we have $y \diamond (y * x) \leqslant x$ and $y * (y \diamond x) \leqslant x$. Thus $x = y \diamond (y * x) = y * (y \diamond x)$. Consequently, $\mathfrak{X}$ satisfies (BC).

(e) $\Longrightarrow$ (b): If $x \leqslant y$, then x, y are in the same branch and, by (e), $x = x \wedge y = y \diamond (y * x) = y * (y \diamond x)$. Therefore, we obtain (b).

(b) $\Longrightarrow$ (a) follows from Proposition 4.5. $\square$

In [4], Dymek obtained an axiomatization of branchwise commutative pseudo-BCI algebras. We give an alternative axiomatization of such algebras.

Theorem 4.10. *An algebra $\mathfrak{X} = (X; *, \diamond, 0)$ of type $(2, 2, 0)$ is a branchwise commutative pseudo-BCH algebra if and only if it satisfies the following identities:*

(A1) $x * 0 = x = x \diamond 0$,
(A2) $(x * y) \diamond z = (x \diamond z) * y$,
(A3) $(x \diamond (x * y)) \diamond y = 0 = (x * (x \diamond y)) * y$,
(A4) $x \diamond (x * y) = y \diamond (y * (x \diamond (x * y)))$,
(A5) $x * (x \diamond y) = y * (y \diamond (x * (x \diamond y)))$.

Proof. If $\mathfrak{X}$ is a branchwise commutative pseudo-BCH algebra, then, obviously, the identities (A1)–(A5) hold for all $x, y \in X$. Conversely, suppose that $\mathfrak{X}$ satisfies (A1)–(A5). Putting $y = 0$ in (A3) and applying (A1), we obtain (pBCH-1). To prove (pBCH-3), let $x * y = y * x = 0$. Using (A1) and (A4), we get

$$x = x \diamond 0 = x \diamond (x * y) = y \diamond (y * (x \diamond (x * y))) = y \diamond (y * x) = y \diamond 0 = y,$$

that is, (pBCH-3) holds in $\mathfrak{X}$. We now prove that

$$x * y = 0 \Longleftrightarrow x \diamond y = 0.$$

If $x * y = 0$, then $(x \diamond 0) \diamond y = 0$ by (A3), and hence $x \diamond y = 0$. Thus $x * y = 0$ implies $x \diamond y = 0$, and analogously, $x \diamond y = 0$ entails $x * y = 0$. Therefore $\mathfrak{X}$ satisfies (pBCH-4), and finally, it is a pseudo-BCH algebra. Moreover, $\mathfrak{X}$ is branchwise commutative by Theorem 4.9. □

Remark. From Theorem 3.11 of [4] we see that the variety of all branchwise commutative pseudo-BCH/BCI algebras is weakly regular.

References

[1] Dudek, W. A., Jun, Y. B., *Pseudo-BCI-algebras*, East Asian Math. J. **24** (2008), 187–190.

[2] Dudek, W. A., Zhang, X., Wang, Y., *Ideals and atoms of BZ-algebras*, Math. Slovaca **59** (2009), 387–404.

[3] Dudek, W. A., Karamdin, B., Bhatti, S. A., *Branches and ideals of weak BCC-algebras*, Algebra Colloquium **18** (Special) (2011), 899–914.

[4] Dymek, G., *On two classes of pseudo-BCI-algebras,* Discuss. Math. Gen. Algebra Appl. **31** (2011), 217–230.

[5] Georgescu, G., Iorgulescu, A., *Pseudo-MV algebras: a noncommutative extension of MV algebras*, in: *The Proc. of the Fourth International Symp. on Economic Informatics*, Bucharest, Romania, May 1999, 961–968.

[6] Georgescu, G., Iorgulescu, A., *Pseudo-BL algebras: a noncommutative extension of BL algebras*, in: *Abstracts of the Fifth International Conference FSTA 2000*, Slovakia, February 2000, 90–92.

[7] Georgescu, G., Iorgulescu, A., *Pseudo-BCK algebras: an extension of BCK algebras*, in: *Proc. of DMTCS'01: Combinatorics, Computability and Logic*, Springer, London, 2001, 97–114.

[8] Hu, Q. P., Li, X., *On BCH-algebras*, Math. Seminar Notes **11** (1983), 313–320.

[9] Imai, Y., Iséki, K., *On axiom systems of propositional calculi XIV*, Proc. Japan Acad. Ser. A Math. Sci. **42** (1966), 19–22.

[10] Iorgulescu A., *Algebras of Logic as BCK-Algebras*, Bucharest 2008.
[11] Iorgulescu, A., *New generalizations of BCI, BCK and Hilbert algebras – Part I*, J. Mult.-Valued Logic Soft Comput. **27** (2016), 353–406.
[12] Iorgulescu, A., *New generalizations of BCI, BCK and Hilbert algebras – Part II*, J. Mult.-Valued Logic Soft Comput. **27** (2016), 407–456.
[13] Iséki, K., *An algebra related with a propositional calculus*, Proc. Japan Acad. Ser. A Math. Sci. **42** (1966), 26–29.
[14] Walendziak, A., *Pseudo-BCH-algebras*, Discuss. Math. Gen. Algebra Appl. **35** (2015), 1–15.
[15] Walendziak, A., *On ideals of pseudo-BCH-algebras*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **70** (2016), 81–91.
[16] Walendziak, A., *Strong ideals and horizontal ideals in pseudo-BCH-algebras*, Ann. Univ. Paedagog. Crac. Stud. Math. **15** (2016), 15–25.
[17] Zhang, X., Ye, R., *BZ-algebra and group*, J. of Mathematical and Physical Sciences **29** (1995), 223–233.
[18] Zhang, X., Wang, Y., Dudek, W. A., *T-ideals in BZ-algebras and T-type BZ-algebras*, Indian J. Pure Appl. Math. **34** (2003), 1559–1570.

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