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UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
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SEVER S. DRAGOMIR

Some new inequalities of Hermite–Hadamard type for GA-convex functions

Abstract. Some new inequalities of Hermite–Hadamard type for GA-convex functions defined on positive intervals are given. Refinements and weighted version of known inequalities are provided. Some applications for special means are also obtained.

1. Introduction. Let $J \subset (0,\infty)$ be an interval; a real-valued function $f : J \to \mathbb{R}$ is said to be *GA-convex* (*concave*) on J (see for instance [2]), if

$$f\left(x^{1-\lambda}y^{\lambda}\right) \le (\ge)\,(1-\lambda)\,f(x) + \lambda f(y) \tag{1.1}$$

for all $x, y \in J$ and $\lambda \in [0,1]$.

Since the condition (1.1) can be written as

$$\begin{aligned} &f \circ \exp\left((1-\lambda)\ln x + \lambda \ln y\right) \\ &\qquad \le (\ge)\,(1-\lambda)\,f \circ \exp(\ln x) + \lambda f \circ \exp(\ln y), \end{aligned} \tag{1.2}$$

then we observe that $f : J \to \mathbb{R}$ is *GA-convex* (concave) on J if and only if $f \circ \exp$ is convex (concave) on $\ln J := \{\ln z, z \in J\}$. If $J = [a,b]$, then $\ln J = [\ln a, \ln b]$.

It is known that the function $f(x) = \ln(1+x)$ is GA-convex on $(0,\infty)$ [2].

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For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [28] that the function $f : (0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is GA-concave on $(0,\infty)$ while the function $g : (0,\infty) \to \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is GA-convex on $(0,\infty)$.

Example 1. If $[a,b] \subset (0,\infty)$ and the function $g : [\ln a, \ln b] \to R$ is convex (concave) on $[\ln a, \ln b]$, then the function $f : [a,b] \to R$, $f(t) = g(\ln t)$ is GA-convex (concave) on $[a,b]$.

We recall that the classical Hermite–Hadamard inequality states that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2} \tag{1.3}$$

for any convex function $f : [a,b] \to \mathbb{R}$.

For related results, see [1]–[6], [8]–[11], [12]–[18] and [20]–[27].

The *identric mean* $I(a,b)$ is defined by

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b) := \frac{b-a}{\ln b - \ln a}$$

where $0 < a < b$.

In [28] the authors obtained the following Hermite–Hadamard type inequality.

Theorem 1. *If $b > a > 0$ and $f : [a,b] \to \mathbb{R}$ is a differentiable GA-convex (concave) function on $[a,b]$, then*

$$\begin{aligned} f(I(a,b)) &\le (\ge) \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge) \frac{b-L(a,b)}{b-a} f(b) + \frac{L(a,b)-a}{b-a} f(a). \end{aligned} \tag{1.4}$$

The differentiability of the function is not necessary in Theorem 1 for the first inequality (1.4) to hold, as shown in [10].

If we take $\lambda = \frac{1}{2}$ in the definition (1.1) of GA-convex (concave) function on $[a, b]$, then we have

$$f\left(\sqrt{ab}\right) \le (\ge) \frac{f(a) + f(b)}{2}. \tag{1.5}$$

The following refinement of (1.5), which is an inequality of Hermite–Hadamard type, holds (see [25] for an extension for GA h-convex functions):

Theorem 2. *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a GA-convex (concave) function on $[a, b]$. Then we have*

$$f\left(\sqrt{ab}\right) \le (\ge) \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \le (\ge) \frac{f(a) + f(b)}{2}. \tag{1.6}$$

Motivated by the above results we provide in the following a refinement of (1.6) for a division of the interval $[a, b]$. We also establish a weighted version that generalizes the inequalities (1.4) and (1.6) and provides upper and lower bounds for the moments

$$\frac{1}{b-a} \int_a^b t^p f(t)\, dt, \text{ for } p \in \mathbb{R} \text{ with } p \ne 0, -1$$

of the GA-convex (concave) function on $[a, b]$.

2. Some refinements. In 1994, [5] (see also [17, p. 22]) we proved the following refinement of Hermite–Hadamard inequality. For the sake of completeness we give here a direct proof that is different from the one in [5].

Lemma 1. *Let $g : [c, d] \to \mathbb{R}$ be a convex function on $[c, d]$. Then for any division $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = d$ with $n \ge 1$ we have the inequalities*

$$\begin{aligned} g\left(\frac{c+d}{2}\right) &\le \frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i)\, g\left(\frac{x_{i+1} + x_i}{2}\right) \\ &\le \frac{1}{d-c} \int_c^d g(x)\, dx \\ &\le \frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \frac{g(x_i) + g(x_{i+1})}{2} \\ &\le \frac{1}{2} [g(c) + g(d)]. \end{aligned} \tag{2.1}$$

Proof. Using the Hermite–Hadamard inequality on the interval $[x_i, x_{i+1}]$, $i \in \{0, \dots, n-1\}$, we have

$$
\begin{aligned}
(x_{i+1} - x_i)\, g\left(\frac{x_{i+1} + x_i}{2}\right) &\le \int_{x_i}^{x_{i+1}} g(x)\, dx \\
&\le (x_{i+1} - x_i)\, \frac{g(x_i) + g(x_{i+1})}{2}
\end{aligned} \tag{2.2}
$$

for any $i \in \{0, \dots, n-1\}$.

Summing in (2.2) over $i \in \{0, \dots, n-1\}$ and dividing by $d - c$, we get the second and the third inequalities in (2.1).

Since for $p_i := \frac{x_{i+1} - x_i}{d - c} \ge 0$ we have $\sum_{i=0}^{n-1} p_i = 1$, then by Jensen's inequality we have

$$
\begin{aligned}
\frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i)\, g\left(\frac{x_{i+1} + x_i}{2}\right) &\ge g\left(\frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left(\frac{x_{i+1} + x_i}{2}\right)\right) \\
&= g\left(\frac{1}{d-c} \sum_{i=0}^{n-1} \left(\frac{x_{i+1}^2 - x_i^2}{2}\right)\right) \\
&= g\left(\frac{d^2 - c^2}{2(d-c)}\right) = g\left(\frac{c+d}{2}\right),
\end{aligned}
$$

which proves the first inequality in (2.1).

For a convex function $g : [c, d] \to \mathbb{R}$ we have

$$
g(x) = g\left(\frac{(d-x)\, c + (x-c)\, d}{d-c}\right) \le \frac{(d-x)\, g(c) + (x-c)\, g(d)}{d-c}
$$

for any $x \in [c, d]$.

This implies that

$$
g(x_i) \le \frac{1}{d-c} \left[(d - x_i)\, g(c) + (x_i - c)\, g(d)\right]
$$

and

$$
g(x_{i+1}) \le \frac{1}{d-c} \left[(d - x_{i+1})\, g(c) + (x_{i+1} - c)\, g(d)\right]
$$

for any $i \in \{0, \dots, n-1\}$.

Therefore,

$$
\begin{aligned}
&\sum_{i=0}^{n-1} (x_{i+1} - x_i)\, \frac{g(x_i) + g(x_{i+1})}{2} \\
&\le \frac{1}{2(d-c)} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \big[(d - x_i) g(c) \\
&\qquad + (x_i - c) g(d) + (d - x_{i+1}) g(c) + (x_{i+1} - c) g(d)\big]
\end{aligned}
$$

$$= \frac{1}{d-c}\left[\left(d\left(d-c\right)-\frac{1}{2}\left(d^2-c^2\right)\right)g\left(c\right)+\left(\frac{1}{2}\left(d^2-c^2\right)-c\left(d-c\right)\right)g\left(d\right)\right]$$
$$= \left(d-c\right)\frac{g\left(c\right)+g\left(d\right)}{2},$$

and the last part of (2.1) is proved. □

Theorem 3. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function on $[a,b]$. Then for any division $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ with $n \geq 1$ we have the inequalities*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \frac{1}{\ln b - \ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1} - \ln t_i\right) f\left(\sqrt{t_i t_{i+1}}\right) \\ &\leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f\left(s\right)}{s}ds \\ &\leq \frac{1}{\ln b - \ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1} - \ln t_i\right)\frac{f\left(t_i\right)+f\left(t_{i+1}\right)}{2} \\ &\leq \frac{1}{2}\left[f\left(a\right)+f\left(b\right)\right]. \end{aligned} \tag{2.3}$$

Proof. If $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a GA-convex function on $[a,b]$, then $f \circ \exp$ is convex on $[\ln a, \ln b]$. Let $c = \ln a$, $d = \ln b$ and the division $x_i = \ln t_i$, $i \in \{0,\ldots,n-1\}$ of the interval $[c,d]$.

If we write the inequality (2.1) for $g = f \circ \exp$ on the interval $[c,d]$ and for the division

$$\ln a = \ln t_0 < \ln t_1 < \cdots < \ln t_{n-1} < \ln t_n = \ln b,$$

we have

$$\begin{aligned} &\left(f\circ\exp\right)\left(\frac{\ln a+\ln b}{2}\right) \\ &\leq \frac{1}{\ln b - \ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1} - \ln t_i\right)\left(f\circ\exp\right)\left(\frac{\ln t_{i+1}+\ln t_i}{2}\right) \\ &\leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b}\left(f\circ\exp\right)\left(x\right)dx \\ &\leq \frac{1}{\ln b - \ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1} - \ln t_i\right)\frac{\left(f\circ\exp\right)\left(\ln t_i\right)+\left(f\circ\exp\right)\left(\ln t_{i+1}\right)}{2} \\ &\leq \frac{1}{2}\left[\left(f\circ\exp\right)\left(\ln a\right)+\left(f\circ\exp\right)\left(\ln b\right)\right], \end{aligned} \tag{2.4}$$

that is equivalent to

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\le \frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1}-\ln t_i\right) f\left(\sqrt{t_i t_{i+1}}\right)\\
&\le \frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left(f\circ\exp\right)(x)\,dx\\
&\le \frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}\left(\ln t_{i+1}-\ln t_i\right)\frac{f(t_i)+f(t_{i+1})}{2}\\
&\le \frac{1}{2}\left[f(a)+f(b)\right].
\end{aligned}
\tag{2.5}
$$

By using the change of variable $\exp(x)=t$, we have $x=\ln t$, $dx=\frac{dt}{t}$ and

$$
\int_{\ln a}^{\ln b}\left(f\circ\exp\right)(x)\,dx=\int_a^b\frac{f(t)}{t}dt
$$

and by (2.5) we get the desired result (2.3). □

Corollary 1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. If $a\le t\le b$, then we have*

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\le \frac{1}{\ln b-\ln a}\left[\left(\ln t-\ln a\right) f\left(\sqrt{at}\right)+\left(\ln b-\ln t\right) f\left(\sqrt{tb}\right)\right]\\
&\le \frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds\\
&\le \frac{1}{2}\left[f(t)+\frac{\left(\ln t-\ln a\right) f(a)+\left(\ln b-\ln t\right) f(b)}{\ln b-\ln a}\right]\\
&\le \frac{1}{2}\left[f(a)+f(b)\right].
\end{aligned}
\tag{2.6}
$$

Remark 1. Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. For a division $0=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n=1$ with $n\ge 1$ of the interval $[0,1]$, consider the division

$$
t_i:=a\left(\frac{b}{a}\right)^{\lambda_i}=a^{1-\lambda_i}b^{\lambda_i},\ \ i\in\{0,\dots,n-1\}
\tag{2.7}
$$

of the interval $[a,b]$.

Observe that

$$
\begin{aligned}
\ln t_{i+1}-\ln t_i &= \ln\left[a\left(\frac{b}{a}\right)^{\lambda_{i+1}}\right]-\ln\left[a\left(\frac{b}{a}\right)^{\lambda_i}\right]\\
&= \ln a+\lambda_{i+1}\ln\left(\frac{b}{a}\right)-\ln a-\lambda_i\ln\left(\frac{b}{a}\right)\\
&= \left(\ln b-\ln a\right)\left(\lambda_{i+1}-\lambda_i\right)
\end{aligned}
$$

and

$$\sqrt{t_i t_{i+1}} = \sqrt{a^{1-\lambda_i} b^{\lambda_i} a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}} = a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}$$

for any $i \in \{0, \dots, n-1\}$.

If we write the inequality (2.3) for the division (2.7) we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) f\left(a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right) \\ &\le \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ &\le \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \frac{f\left(a^{1-\lambda_i} b^{\lambda_i}\right) + f\left(a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}\right)}{2} \\ &\le \frac{1}{2} \left[f(a) + f(b)\right], \end{aligned} \tag{2.8}$$

for any division $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = 1$ with $n \ge 1$ of the interval $[0, 1]$.

If we write the inequality (2.8) for $0 < \lambda < 1$, then we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\le \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\le \frac{1}{2} \left[f\left(a^{1-\lambda} b^{\lambda}\right) + (1-\lambda) f(b) + \lambda f(a)\right] \\ &\le \frac{f(a) + f(b)}{2}. \end{aligned} \tag{2.9}$$

The inequality (2.9) was obtained in [10].

In the following section we establish some weighted Hermite–Hadamard type inequalities for *GA*-convex functions.

3. Weighted inequalities. We have the following weighted inequality:

Theorem 4. *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a GA-convex function on $[a, b]$ and $w : [a, b] \to [0, \infty)$ an integrable function on $[a, b]$ with $\int_a^b w(t)\, dt > 0$, then*

$$\begin{aligned} f\left(\exp\left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t)\, dt}\right)\right) &\le \frac{\int_a^b w(t) f(t)\, dt}{\int_a^b w(t)\, dt} \\ &\le \frac{\left(\ln b - \frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt}\right) f(a) + \left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt} - \ln a\right) f(b)}{\ln b - \ln a}. \end{aligned} \tag{3.1}$$

Proof. Observe that for $t \in [a,b]$ we have

$$\ln t = \frac{(\ln b - \ln t)\ln a + (\ln t - \ln a)\ln b}{\ln b - \ln a}.$$

By the convexity of $f \circ \exp$ we have

$$\begin{aligned} f(t) &= (f \circ \exp)(\ln t) \\ &= (f \circ \exp)\left(\frac{(\ln b - \ln t)\ln a + (\ln t - \ln a)\ln b}{\ln b - \ln a}\right) \\ &\le \frac{\ln b - \ln t}{\ln b - \ln a} f(a) + \frac{\ln t - \ln a}{\ln b - \ln a} f(b) \end{aligned} \tag{3.2}$$

for any $t \in [a,b]$.

If we multiply (3.2) by $w(t) \ge 0$, $t \in [a,b]$ and integrate over t on $[a,b]$, then we get

$$\int_a^b f(t)\, w(t)\, dt \le \frac{\ln b \int_a^b w(t)\, dt - \int_a^b w(t) \ln t dt}{\ln b - \ln a} f(a) + \frac{\int_a^b w(t)\ln t dt - \ln a \int_a^b w(t)\, dt}{\ln b - \ln a} f(b)$$

and the second inequality in (3.1) is proved.

By Jensen's inequality we have

$$\frac{\int_a^b w(t) f(t)\, dt}{\int_a^b w(t)\, dt} = \frac{\int_a^b w(t)(f \circ \exp)(\ln t)\, dt}{\int_a^b w(t)\, dt} \ge (f \circ \exp)\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\, dt}\right)$$

and the first part of (3.1) is proved. □

Corollary 2 (see Theorem 2). *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function on $[a,b]$, then*

$$f\left(\sqrt{ab}\right) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t} dt \le \frac{f(a) + f(b)}{2}. \tag{3.3}$$

Proof. If we take $w(t) = \frac{1}{t}$ in (3.1), then we have

$$\begin{aligned} f\left(\exp\left(\frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt}\right)\right) &\le \frac{\int_a^b \frac{f(t)}{t} dt}{\int_a^b \frac{1}{t} dt} \\ &\le \frac{\left(\ln b - \frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt}\right) f(a) + \left(\frac{\int_a^b \frac{\ln t}{t} dt}{\int_a^b \frac{1}{t} dt} - \ln a\right) f(b)}{\ln b - \ln a}. \end{aligned} \tag{3.4}$$

Since

$$\int_a^b \frac{\ln t}{t} dt = \frac{1}{2}\left[(\ln b)^2 - (\ln a)^2\right], \quad \int_a^b \frac{1}{t} dt = \ln b - \ln a,$$

then we get from (3.4)

$$f\left(\exp\left(\frac{\ln b+\ln a}{2}\right)\right) \leq \frac{\int_a^b \frac{f(t)}{t}dt}{\ln b-\ln a}$$
$$\leq \frac{\left(\ln b-\frac{\ln b+\ln a}{2}\right)f(a)+\left(\frac{\ln b+\ln a}{2}-\ln a\right)f(b)}{\ln b-\ln a}$$

and the inequality (3.3) is proved. □

Define the *p-logarithmic mean* as

$$L_p(a,b) := \left(\frac{1}{p+1}\frac{b^{p+1}-a^{p+1}}{b-a}\right)^{1/p}, \ p \neq -1,\ 0$$

for $0<a<b$.

Corollary 3. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$, then for any $p\in\mathbb{R}$ with $p\neq 0,-1$ we have*

$$\begin{aligned} &L_p^p(a,b)\, f\left(\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right) \\ &\leq \frac{1}{b-a}\int_a^b t^p f(t)\,dt \\ &\leq \frac{\left(L\left(a^{p+1},b^{p+1}\right)-a^{p+1}\right)f(a)+\left(b^{p+1}-L\left(a^{p+1},b^{p+1}\right)\right)f(b)}{(p+1)(b-a)}. \end{aligned} \tag{3.5}$$

If $p=0$, then we have

$$\begin{aligned} f(I(a,b)) &\leq \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{(L(a,b)-a)f(a)+(b-L(a,b))f(b)}{b-a}. \end{aligned} \tag{3.6}$$

Proof. If we take $w(t)=t^p$, with $p\neq 0,-1$ in (3.1), then we have

$$\begin{aligned} &f\left(\exp\left(\frac{\int_a^b t^p\ln t dt}{\int_a^b t^p dt}\right)\right) \\ &\leq \frac{\int_a^b t^p f(t)\,dt}{\int_a^b t^p dt} \\ &\leq \frac{\left(\ln b-\frac{\int_a^b t^p\ln t dt}{\int_a^b t^p dt}\right)f(a)+\left(\frac{\int_a^b t^p\ln t dt}{\int_a^b t^p dt}-\ln a\right)f(b)}{\ln b-\ln a}. \end{aligned} \tag{3.7}$$

We have

$$\int_a^b t^p dt = \frac{1}{p+1}\left(b^{p+1}-a^{p+1}\right)$$

and

$$\begin{aligned}\int_a^b t^p \ln t dt &= \frac{1}{p+1}\int_a^b \ln t dt^{p+1} \\ &= \frac{1}{p+1}\left[t^{p+1}\ln t\Big|_a^b - \int_a^b t^p dt\right] \\ &= \frac{1}{p+1}\left(b^{p+1}\ln b - a^{p+1}\ln a - \frac{1}{p+1}\left(b^{p+1}-a^{p+1}\right)\right).\end{aligned}$$

Then

$$\begin{aligned}\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} &= \frac{\frac{1}{p+1}\left(b^{p+1}\ln b - a^{p+1}\ln a - \frac{1}{p+1}\left(b^{p+1}-a^{p+1}\right)\right)}{\frac{1}{p+1}\left(b^{p+1}-a^{p+1}\right)} \\ &= \frac{1}{p+1}\ln I\left(a^{p+1},b^{p+1}\right) = \ln\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}},\end{aligned}$$

$$\begin{aligned}&\ln b - \frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} \\ &= \ln b - \frac{1}{p+1}\left[\frac{b^{p+1}\ln b^{p+1} - a^{p+1}\ln a^{p+1}}{b^{p+1}-a^{p+1}} - 1\right] \\ &= \frac{1}{p+1}\left(1 - \frac{\ln b^{p+1} - \ln a^{p+1}}{b^{p+1}-a^{p+1}}a^{p+1}\right) = \frac{1}{p+1}\left(1 - \frac{a^{p+1}}{L\left(a^{p+1},b^{p+1}\right)}\right) \\ &= \frac{1}{p+1}\frac{L\left(a^{p+1},b^{p+1}\right) - a^{p+1}}{L\left(a^{p+1},b^{p+1}\right)}\end{aligned}$$

and, similarly,

$$\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} - \ln a = \frac{1}{p+1}\frac{b^{p+1} - L\left(a^{p+1},b^{p+1}\right)}{L\left(a^{p+1},b^{p+1}\right)}.$$

From (3.7) we then have

$$\begin{aligned}&f\left(\exp\left(\ln\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right)\right) \\ &\le \frac{\frac{1}{b-a}\int_a^b t^p f(t)\,dt}{\frac{1}{b-a}\int_a^b t^p dt} \\ &\le \frac{\left(\frac{1}{p+1}\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L\left(a^{p+1},b^{p+1}\right)}\right)f(a) + \left(\frac{1}{p+1}\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L\left(a^{p+1},b^{p+1}\right)}\right)f(b)}{\ln b - \ln a},\end{aligned}$$

i.e.

$$
\begin{aligned}
& f\left(\left[I\left(a^{p+1}, b^{p+1}\right)\right]^{\frac{1}{p+1}}\right) \\
& \leq \frac{\frac{1}{b-a}\int_a^b t^p f(t)\,dt}{L_p^p(a,b)} \\
& \leq \frac{\left(\frac{1}{p+1}\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L(a^{p+1},b^{p+1})}\right) f(a) + \left(\frac{1}{p+1}\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L(a^{p+1},b^{p+1})}\right) f(b)}{\ln b - \ln a} \\
& = \frac{\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L(a^{p+1},b^{p+1})} f(a) + \frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L(a^{p+1},b^{p+1})} f(b)}{\ln b^{p+1} - \ln a^{p+1}}.
\end{aligned}
$$

By multiplying this inequality with $L_p^p(a,b)$, we get

$$
\begin{aligned}
& L_p^p(a,b)\, f\left(\left[I\left(a^{p+1}, b^{p+1}\right)\right]^{\frac{1}{p+1}}\right) \\
& \leq \frac{1}{b-a}\int_a^b t^p f(t)\,dt \\
& \leq \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \frac{\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L(a^{p+1},b^{p+1})} f(a) + \frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L(a^{p+1},b^{p+1})} f(b)}{\ln b^{p+1} - \ln a^{p+1}} \\
& = \frac{\left(L\left(a^{p+1}, b^{p+1}\right) - a^{p+1}\right) f(a) + \left(b^{p+1} - L\left(a^{p+1}, b^{p+1}\right)\right) f(b)}{(p+1)(b-a)}.
\end{aligned}
$$

If we perform the calculations in the above inequalities for $p = 0$, we get the desired inequality (3.6). We omit the details. □

Remark 2. If we take $p = 1$ in (3.5), then we get

$$
\begin{aligned}
& f\left(\sqrt{I\left(a^2, b^2\right)}\right) \leq \frac{1}{b-a}\int_a^b t f(t)\,dt \\
& \leq \frac{\left(A(a,b)\,L(a,b) - a^2\right) f(a) + \left(b^2 - A(a,b)\,L(a,b)\right) f(b)}{2(b-a)}.
\end{aligned} \tag{3.8}
$$

4. Applications. Let $q \neq 0$ and consider the convex function $g(t) = \exp(qt)$, $t \in \mathbb{R}$. Then the function $f_q : (0,\infty) \to \mathbb{R}$, $f_q(t) = g(\ln t) = \exp(q \ln t) = t^q$ is a GA-convex function on $(0,\infty)$. We observe that for $0 < a < b$ we have

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b t^q dt &= \begin{cases} \frac{1}{q+1}\frac{b^{q+1}-a^{q+1}}{b-a}, & q \neq -1 \\ \\ \frac{\ln b - \ln a}{b-a}, & q = -1 \end{cases} \\
&= \begin{cases} L_q^q(a,b), & q \neq -1 \\ \\ L^{-1}(a,b), & q = -1, \end{cases}
\end{aligned}
$$

where $L_q(a,b)$ $(q \neq -1)$ is the q-*Logarithmic mean* and L is the logarithmic mean defined in the introduction.

If we write the inequality (2.8) for the GA-convex function on $(0,\infty)$, $f_q : (0,\infty) \to \mathbb{R}$, $f_q(t) = t^q$, $q \neq 0$, then we get the following result:

Proposition 1. *For any division* $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 1$ *with* $n \geq 1$ *of the interval* $[0,1]$,

$$\begin{aligned} G^q(a,b) &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i)\, G^q\left(a^{2-\lambda_i-\lambda_{i+1}}, b^{\lambda_i+\lambda_{i+1}}\right) \\ &\leq L(a,b)\, L_{q-1}^{q-1}(a,b) \\ &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i)\, \frac{G^q\left(a^{2(1-\lambda_i)}, b^{2\lambda_i}\right) + G^q\left(a^{2(1-\lambda_{i+1})}, b^{2\lambda_{i+1}}\right)}{2} \\ &\leq A(a^q, b^q)\,. \end{aligned} \tag{4.1}$$

If we write the inequality (3.5) for the GA-convex function on $(0,\infty)$, $f_q : (0,\infty) \to \mathbb{R}$, $f_q(t) = t^q$, $q \neq 0$, then:

Proposition 2. *If* $0 < a < b$ *and* $p,\ q \in \mathbb{R}$ *with* $p \neq 0, -1,\ q \neq 0$, *then*

$$\begin{aligned} &L_p^p(a,b)\left[I\left(a^{p+1}, b^{p+1}\right)\right]^{\frac{q}{p+1}} \\ &\leq L_{p+q}^{p+q}(a,b) \\ &\leq \frac{\left(L\left(a^{p+1}, b^{p+1}\right) - a^{p+1}\right) a^q + \left(b^{p+1} - L\left(a^{p+1}, b^{p+1}\right)\right) b^q}{(p+1)(b-a)}. \end{aligned} \tag{4.2}$$

From (3.6) and (3.8) we get:

Proposition 3. *If* $0 < a < b$ *and* $q \in \mathbb{R}$ *with* $q \neq 0$, *then*

$$I^q(a,b) \leq L_q^q(a,b) \leq \frac{(L(a,b) - a)\, a^q + (b - L(a,b))\, b^q}{b-a}, \tag{4.3}$$

and

$$\begin{aligned} I^{q/2}\left(a^2, b^2\right) &\leq L_{q+1}^{q+1}(a,b) \\ &\leq \frac{\left(A(a,b)\, L(a,b) - a^2\right) a^q + \left(b^2 - A(a,b)\, L(a,b)\right) b^q}{2(b-a)}. \end{aligned} \tag{4.4}$$

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S. S. Dragomir
College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001
Australia
e-mail: sever.dragomir@vu.edu.au
url: http://rgmia.org/dragomir

DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences
School of Computer Science and Applied Mathematics
University of the Witwatersrand
Private Bag 3, Johannesburg 2050
South Africa

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