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On the Courant bracket on couples of vector fields and p-forms

Abstract. If $m \geq p+1 \geq 2$ (or $m = p \geq 3$), all natural bilinear operators A transforming pairs of couples of vector fields and p-forms on m-manifolds M into couples of vector fields and p-forms on M are described. It is observed that any natural skew-symmetric bilinear operator A as above coincides with the generalized Courant bracket up to three (two, respectively) real constants.

1. Introduction. In the whole paper the word "bilinear" means "bilinear over $\mathbb{R}$".

Let $\mathcal{M}f_m$ be the category of m-dimensional $\mathcal{C}^\infty$ manifolds and their embeddings.

The "doubled" tangent bundle $T \oplus T^*$ over $\mathcal{M}f_m$ is full of interest because of the non-degenerate symmetric bilinear form and the Courant bracket, see [2]. The non-degenerate symmetric bilinear form and the Courant bracket on $T \oplus T^*$ are involved in the definitions of Dirac and generalized complex structures, see e.g. [2, 6, 7]. Such structures have applications in the high energy physics, see e.g. [1]. That is why, in [4], we studied brackets similar to the Courant one.

The Courant bracket can be generalized to the one on $T \oplus \bigwedge^p T^*$, see e.g. [7]. That is why, in the present note, we study all $\mathcal{M}f_m$-natural bilinear

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operators

$$A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$$

transforming pairs of couples $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^p(M)$ $(i = 1, 2)$ of vector fields and p-forms on m-manifolds M into couples $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \mathcal{X}(M) \oplus \Omega^p(M)$ of vector fields and p-forms on M.

Roughly speaking, in the present note, we deduce that if $m \geq p+1 \geq 2$ (or $m = p \geq 3$, respectively), then any $\mathcal{M}f_m$-natural skew-symmetric bilinear operator A as above coincides with the generalized Courant bracket up to three (or two, respectively) real constants.

Some linear natural operators on vector fields, forms and some other tensor fields have been studied in many papers, see e.g. [3, 5, 9, 10], etc.

From now on, (x^i) $(i = 1, \dots, m)$ denote the usual coordinates on $\mathbb{R}^m$ and $\partial_i = \frac{\partial}{\partial x^i}$ are the canonical vector fields on $\mathbb{R}^m$.

2. The basic notions.

Definition 2.1. An $\mathcal{M}f_m$-natural bilinear operator $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$ is an $\mathcal{M}f_m$-invariant family of bilinear operators

$$A : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \mathcal{X}(M) \oplus \Omega^p(M)$$

for m-dimensional manifolds M, where $\mathcal{X}(M)$ is the space of vector fields on M and $\Omega^p(M)$ is the space of p-forms on M.

Remark 2.2. We recall that the $\mathcal{M}f_m$-invariance of A means that if

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M))$$

and

$$(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2) \in (\mathcal{X}(\overline{M}) \oplus \Omega^p(\overline{M})) \times (\mathcal{X}(\overline{M}) \oplus \Omega^p(\overline{M}))$$

are φ-related by an $\mathcal{M}f_m$-map $\varphi : M \to \overline{M}$ (i.e. $\overline{X}^i \circ \varphi = T\varphi \circ X^i$ and $\overline{\omega}^i \circ \varphi = \bigwedge^p T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

Definition 2.3. An $\mathcal{M}f_m$-natural bilinear operator $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T$ is an $\mathcal{M}f_m$-invariant family of bilinear operators

$$A : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \mathcal{X}(M)$$

for m-manifolds M.

Definition 2.4. An $\mathcal{M}f_m$-natural bilinear operator $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow \bigwedge^p T^*$ is an $\mathcal{M}f_m$-invariant family of bilinear operators

$$A : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \Omega^p(M)$$

for m-manifolds M.

Remark 2.5. By the multi-linear Peetre theorem, see [8], any $\mathcal{M}f_m$-natural bilinear operator A (as above) is of finite order. It means that there is a finite number r such that we have the following implication

$$(j^r_x X_i = j^r_x \overline{X}_i, j^r_x \omega_i = j^r_x \overline{\omega}_i, i = 1, 2) \Rightarrow A(X_1 \oplus \omega_1, X_2 \oplus \omega_2)_{|x} = A(\overline{X}_1 \oplus \overline{\omega}_1, \overline{X}_2 \oplus \overline{\omega}_2)_{|x}$$

Remark 2.6. We say that an operator A is regular if it transforms smoothly parametrized families of objects into smoothly parametrized families. One can show that $\mathcal{M}f_m$-natural bilinear operators are regular because of the Peetre theorem.

Definition 2.7. An $\mathcal{M}f_m$-natural operator $B : T \oplus T^{(0,0)} \rightsquigarrow \bigwedge^p T^*$ is an $\mathcal{M}f_m$-invariant family of regular (not necessarily bilinear) operators

$$B : \mathcal{X}(M) \oplus \mathcal{C}^\infty(M) \to \Omega^p(M)$$

for m-manifolds M, where $\mathcal{C}^\infty(M)$ is the space of smooth maps $M \to \mathbb{R}$.

The most interesting $\mathcal{M}f_m$-natural bilinear operator $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow (T \oplus \bigwedge^p T^*)$ is the generalized Courant bracket.

Example 2.8 ([7]). The generalized Courant bracket is given by

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_C = [X^1, X^2] \oplus \Big(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1 + \frac{1}{2} d(i_{X^2}\omega^1 - i_{X^1}\omega^2)\Big)$$

for any $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^p(M)$, $i = 1, 2$, where d is the usual differentiation, $\mathcal{L}$ is the Lie derivative, i is the usual inner differentiation and $[-,-]$ is the usual bracket on vector fields. For $p = 1$ we get the usual Courant bracket as in [2].

Remark 2.9. If $m = p$, we have $\mathcal{L}_X\omega = di_X\omega + i_X d\omega = di_X\omega$ for any vector field X and any m-form ω on an m-manifold M as $d\omega = 0$. Consequently, if $m = p$ the generalized Courant bracket satisfies $[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_C = [X^1, X^2] \oplus \frac{1}{2}(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1)$ for any $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^m(M)$, $i = 1, 2$.

3. The main result. The main result of the present note is the following classification theorem.

Theorem 3.1. *If $m \geq p + 1 \geq 2$ (or $m = p \geq 3$, respectively), any $\mathcal{M}f_m$-natural bilinear operator $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d(i_{X^2}\omega^1) + c_2 d(i_{X^1}\omega^2))$$

for uniquely determined by A real numbers a, b_1, b_2, c_1, c_2 (or a, b_1, b_2, c_1, c_2 with $c_1 = c_2 = 0$, respectively).

Proof. Theorem 3.1 is an immediate consequence of Propositions 3.2 and 3.4. $\square$

Proposition 3.2. *If $m \geq p \geq 1$, any $\mathcal{M}f_m$-natural bilinear operator $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2]$$

for a (uniquely determined by A) real number a.

Proof. For $p = 1$, our Proposition 3.2 is exactly Proposition 4.1 in [4]. (The proof of Proposition 4.1 in [4] works for $m = 1$, too.)

Let A be a $\mathcal{M}f_m$-natural bilinear operator in question. We define new $\mathcal{M}f_m$-natural bilinear operator $\tilde{A} : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$ by $\tilde{A}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = A(X^1 \oplus 0, X^2 \oplus 0)$. By our proposition for $p = 1$, there is a real number a such that $\tilde{A} = a[-,-]$, i.e.

$$A(X^1 \oplus 0, X^2 \oplus 0) = a[X^1, X^2]$$

for any m-manifold M and any vector fields X^1, X^2 on M.

Our operator A is determined by the values $\langle A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \eta\rangle \in \mathbb{R}$ for all $X^i \oplus \omega^i \in \mathcal{X}(\mathbb{R}^m) \oplus \Omega^p(\mathbb{R}^m)$, $\eta \in T_0^*\mathbb{R}^m$, $i = 1, 2$. Moreover, by the invariance and the regularity of A and the Frobenius theorem we may additionally assume that $X^1 = \partial_1$ and $\eta = \eta^o = d_0x^1$. In other words, A is determined by the values

$$\langle A(\partial_1 \oplus \omega^1, X \oplus \omega^2)_{|0}, \eta^o\rangle \in \mathbb{R}$$

for all $X \in \mathcal{X}(\mathbb{R}^m)$, $\omega^i \in \Omega^p(\mathbb{R}^m)$, $i = 1, 2$. Using the invariance of A with respect to the homotheties and the bilinearity of A, we have the homogeneity condition

$$\begin{aligned} &\left\langle A\Big(\partial_1 \oplus t\Big(\frac{1}{t}id\Big)_* \omega^1, t\Big(\frac{1}{t}id\Big)_* X \oplus t\Big(\frac{1}{t}id\Big)_* \omega^2\Big)_{|0}, \eta^o\right\rangle \\ &\qquad = t\langle A(\partial_1 \oplus \omega^1, X \oplus \omega^2)_{|0}, \eta^o\rangle. \end{aligned}$$

So, by the homogeneous function theorem, since A is of finite order and regular, the value $\langle A(\partial_1 \oplus \omega^1, X \oplus \omega^2)_{|0}, \eta^o\rangle$ depends on $j_0^1 X$, only. Then A is determined by the values $\langle A(\partial_1 \oplus 0, X \oplus 0)_{|0}, \eta^o\rangle$, i.e. A is determined by the number a.

Consequently, the vector space of all $\mathcal{M}f_m$-natural bilinear operators $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T$ is not more than 1-dimensional. On the other hand, we have the $\mathcal{M}f_m$-natural bilinear operator A_o (in question) given by $A_o(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [X^1, X^2]$. The proof of Proposition 3.2 is complete. □

Lemma 3.3. *Let $B : T \oplus T^{(0,0)} \rightsquigarrow \bigwedge^p T^*$ be an $\mathcal{M}f_m$-natural operator satisfying*

$$\begin{aligned} &B(tX \oplus f) = t^2 B(X \oplus f) = B(X \oplus t^2 f), \\ &B(X \oplus (f + f_1)) = B(X \oplus f) + B(X \oplus f_1). \end{aligned}$$

If $m \geq p + 1 \geq 3$ (or $m = p \geq 3$, respectively), then $B = 0$.

Proof. By the classical Peetre theorem (since B is linear in f), B is of finite order in f, i.e. for any m-manifold M, any point $x \in M$ and any vector field $X \in \mathcal{X}(M)$ there is a natural number r such that for any $f, \overline{f} \in \mathcal{C}^\infty(M)$ from $j^r_x f = j^r_x \overline{f}$ it follows $B(X,f)_{|_x} = B(X,\overline{f})_{|_x}$. Clearly, B is determined by the values $\langle B(X \oplus f)_{|_0}, v\rangle \in \mathbb{R}$ for $X \in \mathcal{X}(\mathbb{R}^m)$, $f \in \mathcal{C}^\infty(M)$, $v \in \bigwedge^p T_0\mathbb{R}^m$. If $m \geq p+1 \geq 2$, by the regularity of B, we may assume that $X_{|_0} \wedge v \neq 0$, and then by the invariance of B and the Frobenius theorem, we may assume that $X = \partial_1$ and $v = v^o = \partial_{2|_0} \wedge \cdots \wedge \partial_{p+1|0}$, i.e. B is determined by the values

$$\langle B(\partial_1 \oplus f)_{|_0}, v^o\rangle \in \mathbb{R}$$

for all $f \in \mathcal{C}^\infty(\mathbb{R}^m)$. (If $m = p \geq 1$ we may almost the same assume that B is determined by the values $\langle B(\partial_1 \oplus f)_{|_0}, v^{o,o}\rangle \in \mathbb{R}$ for all $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, where $v^{o,o} = \partial_{1|0} \wedge \cdots \wedge \partial_{m|0}$.) Since B is of finite order in f, we may assume that f is a polynomial. Now, by the invariance of B with respect to the diffeomorphisms $(t_1x^1, \ldots, t_mx^m)$, $t_l \in \mathbb{R}_+$, $l = 1, \ldots, m$ and the conditions of B, we derive that if $m \geq p+1 \geq 2$ (or $m = p \geq 1$, respectively), then $\langle B(\partial_1 \oplus f)_{|_0}, v^o\rangle$ (or $\langle B(\partial_1 \oplus f)_{|_0}, v^{o,o}\rangle$, respectively) is determined by $\langle B(\partial_1 \oplus (x^1)^2x^2 \ldots x^{p+1})_{|_0}, v^o\rangle$ (or of $\langle B(\partial_1 \oplus (x^1)^3x^2 \ldots x^m)_{|_0}, v^{o,o}\rangle$, respectively).

Then if $m \geq p+1 \geq 2$ (or $m = p \geq 1$, respectively), B is determined by the values $B(\partial_1, (x^1)^2 g)_{|_0}$ (or $B(\partial_1, (x^1)^3 g)_{|_0}$, respectively) for all $g = g(x^2, \ldots, x^m)$. By the invariance of B with respect to the diffeomorphisms of the form $id_{\mathbb{R}} \times \psi$ we may assume that $g = 1$ or $g = x^2$. Then if $m \geq p+1 \geq 3$ (or $m = p \geq 3$, respectively), by the invariance of B with respect to the homotheties and the assumptions on B, we deduce $B(\partial_1, (x^1)^2 g)_{|_0} = 0$ (or $B(\partial_1, (x^1)^3 g)_{|_0} = 0$, respectively). Consequently, $B = 0$ if $m \geq p+1 \geq 3$ (or $m = p \geq 3$, respectively). The proof of Lemma 3.3 is complete. □

Proposition 3.4. *Let $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow \bigwedge^p T^*$ be an $\mathcal{M}f_m$-natural bilinear operator. If $m \geq p+1 \geq 2$ (or $m = p \geq 3$, respectively), then A is the linear combination with real coefficients of the $\mathcal{M}f_m$-natural bilinear operators $A^{<j>} : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow \bigwedge^p T^*$ for $j = 1, \ldots, 4$ (or $j = 1, 2$) given by*

$$A^{<1>}(\rho^1, \rho^2) = \mathcal{L}_{X^2}\omega^1, \;\; A^{<2>}(\rho^1, \rho^2) = \mathcal{L}_{X^1}\omega^2,$$

$$A^{<3>}(\rho^1, \rho^2) = \frac{1}{2}d(i_{X^2}\omega^1 + i_{X^1}\omega^2), \;\; A^{<4>}(\rho^1, \rho^2) = \frac{1}{2}d(i_{X^2}\omega^1 - i_{X^1}\omega^2),$$

where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$.

Proof. Clearly, A is determined by the values

$$\langle A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|_0}, v\rangle \in \mathbb{R}$$

for all $X^1, X^2 \in \mathcal{X}(\mathbb{R}^m)$, $\omega^1, \omega^2 \in \Omega^p(\mathbb{R}^m)$, $v \in \bigwedge^p T_0\mathbb{R}^m$. Consequently, using the bilinearity of A, A is determined by the values

$$\langle A(0 \oplus \omega^1, 0 \oplus \omega^2)_{|_0}, v\rangle, \;\; \langle A(0 \oplus \omega^1, X^2 \oplus 0)_{|_0}, v\rangle,$$

$$\langle A(X^1 \oplus 0, 0 \oplus \omega^2)_{|0}, v\rangle,\ \langle A(X^1 \oplus 0, X^2 \oplus 0)_{|0}, v\rangle$$

for all $X^1, X^2 \in \mathcal{X}(\mathbb{R}^m)$, $\omega^1, \omega^2 \in \Omega^p(\mathbb{R}^m)$, $v \in \bigwedge^p T_0\mathbb{R}^m$.

Using the invariance of A with respect to the homotheties and the bilinearity of A and then applying the homogeneous function theorem, we easily deduce that

$$\langle A(0 \oplus \omega^1, 0 \oplus \omega^2)_{|0}, v\rangle = 0.$$

Similarly, $\langle A(X^1 \oplus 0, 0 \oplus \omega^2)_{|0}, v\rangle$ depends on $j^1_0\omega^2$ and $j^1_0 X^1$ only.

If $m \geq p+1 \geq 2$, by the regularity of A we may assume that $X^1_{|0} \wedge v \neq 0$, and then by the Frobenius theorem and by the invariance we may assume that $X^1 = \partial_1$ and $v = v^o = \partial_{2|0} \wedge \cdots \wedge \partial_{p+1|0}$. (If $m = p \geq 1$, we may assume that $X^1 = \partial_1$ and $v = v^{o,o} = \partial_{1|0} \wedge \cdots \wedge \partial_{m|0}$.)

Then if $m \geq p+1 \geq 2$ (or $m = p \geq 1$, respectively), using the invariance of A with respect to $(t_1x^1, \dots, t_m x^m)$ for $t_l \in \mathbb{R}_+$, $l = 1, \dots, m$, we may assume that $\langle A(X^1 \oplus 0, 0 \oplus \omega^2)_{|0}, v\rangle$ is determined by the real numbers

$$c_1 := \langle A(\partial_1 \oplus 0, 0 \oplus x^1 dx^2 \wedge \cdots \wedge dx^{p+1})_{|0}, v^o\rangle$$

and

$$c_3 := \langle A(\partial_1 \oplus 0, 0 \oplus x^2 dx^1 \wedge dx^3 \wedge \cdots \wedge dx^{p+1})_{|0}, v^o\rangle$$

(or $\tilde{c}_1 := \langle A(\partial_1 \oplus 0, 0 \oplus x^1 dx^1 \wedge \cdots \wedge dx^m)_{|0}, v^{o,o}\rangle$, respectively).

By the similar arguments if $m \geq p+1 \geq 2$ (or $m = p \geq 1$, respectively) we may assume that $\langle A(0 \oplus \omega^1, X^2 \oplus 0)_{|0}, v\rangle$ is determined by the real numbers

$$c_2 := \langle A(0 \oplus x^1 dx^2 \wedge \cdots \wedge dx^{p+1}, \partial_1 \oplus 0)_{|0}, v^o\rangle$$

and

$$c_4 := \langle A(0 \oplus x^2 dx^1 \wedge dx^3 \wedge \cdots \wedge dx^{p+1}, \partial_1 \oplus 0)_{|0}, v^o\rangle$$

(or $\tilde{c}_2 := \langle A(0 \oplus x^1 dx^1 \wedge \cdots \wedge dx^m), \partial_1 \oplus 0)_{|0}, v^{o,o}\rangle$, respectively).

Similarly, if $m \geq p+1 \geq 2$ (or $m = p \geq 1$, respectively) we may assume that $\langle A(X^1 \oplus 0, X^2 \oplus 0)_{|0}, v\rangle$ is determined by the real numbers

$$d_k := \langle A(\partial_1 \oplus 0, x^1x^2 \dots x^{p+1}x^k\partial_k \oplus 0)_{|0}, v^o\rangle,\ k = 1, \dots, m$$

(or $\tilde{d}_k := \langle A(\partial_1 \oplus 0, (x^1)^2x^2 \dots x^m x^k \partial_k \oplus 0)_{|0}, v^{o,o}\rangle$, $k = 1, \dots, m$, respectively).

The above facts imply that if $m \geq p+1 \geq 2$ (or $m = p \geq 1$, respectively), then A is determined by the real numbers d_k (or $\tilde{d}_k$, respectively) together with

$$b_1 := c_2,\ b_2 := c_1,\ b_3 := c_3 + c_4,\ b_4 := c_4 - c_3$$

(or $\tilde{b}_1 := \tilde{c}_2$, $\tilde{b}_2 := \tilde{c}_1$, respectively).

We prove that if $m \geq p+1 \geq 2$ (or $m = p \geq 3$, respectively), then $A = \sum_{j=1}^4 b_j A^{<j>}$ (or $A = \sum_{j=1}^2 \tilde{b}_j A^{<j>}$, respectively).

If $m \geq p+1 \geq 2$ ($m = p \geq 1$, respectively), replacing A by $A - \sum_{j=1}^4 b_j A^{<j>}$ (or $A - \sum_{j=1}^2 \tilde{b}_j A^{<j>}$, respectively), we may assume that

$b_1 = b_2 = b_3 = b_4 = 0$ (or $\tilde{b}_1 = \tilde{b}_2 = 0$, respectively), i.e. we may assume that A is determined by the values d_k (or $\tilde{d}_k$, respectively), i.e. we may assume that A is determined by the value

$$\langle A(\partial_1 \oplus 0, (x^1)^2 x^2 \dots x^{p+1}\partial_1 \oplus 0)_{|0}, v^o\rangle \in \mathbb{R}$$

(or $\langle A(\partial_1 \oplus 0, (x^1)^3 x^2 \dots x^m \partial_1 \oplus 0)_{|0}, v^{o,o}\rangle \in \mathbb{R}$, respectively) together with the values

$$A(\partial_1 \oplus 0, x^1 Y \oplus 0)_{|0} \in \bigwedge^p T_0^*\mathbb{R}^m$$

(or $A(\partial_1 \oplus 0, (x^1)^2 Y \oplus 0)_{|0} \in \bigwedge^m T_0^*\mathbb{R}^m$, respectively) for all vector fields $Y \in \mathcal{X}(\mathbb{R}^{m-1})$ (depending on $x^2, \dots, x^m$). Next, by the regularity of A, we may assume that $Y_{|0} \neq 0$, and then, by the invariance of A with respect to that local diffeomorphisms of the form $id_{\mathbb{R}} \times \psi(x^2, \dots, x^m)$ and the Frobenius theorem, we may assume that $Y = \partial_2$. But $A(\partial_1 \oplus 0, x^1\partial_2 \oplus 0)_{|0} = 0$ (or $A(\partial_1 \oplus 0, (x^1)^2\partial_2 \oplus 0)_{|0} = 0$, respectively) because of the invariance of A with respect to the homotheties. Consequently, A is determined by the $\mathcal{M}f_m$-natural operator $B : T \oplus T^{(0,0)} \rightsquigarrow \bigwedge^p T^*$ given by $B(X \oplus f) := A(X \oplus 0, fX \oplus 0)$, $M \in obj(\mathcal{M}f_m)$, $X \in \mathcal{X}(M)$, $f \in \mathcal{C}^\infty(M)$.

Clearly, B satisfies the assumptions of Lemma 3.3. Then $B = 0$ if $m \geq p+1 \geq 3$ ($m = p \geq 3$, respectively). It means that if $m \geq p+1 \geq 3$ (or $m = p \geq 3$, respectively), then $A = \sum_{j=1}^4 b_j A^{<j>}$ (or $A = \sum_{j=1}^2 \tilde{b}_j A^{<j>}$, respectively), where the numbers b_j (or $\tilde{b}_j$, respectively) are defined above.

If $m \geq p+1 = 2$, our proposition is exactly Proposition 6.1 in [4]. □

From Theorem 3.1 it follows

Corollary 3.5. *If $m \geq p+1 \geq 2$ (or $m = p \geq 3$, respectively), any $\mathcal{M}f_m$-natural skew-symmetric bilinear operator $A : (T \otimes \bigwedge^p T^*) \oplus (T \otimes \bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) + cd(i_{X^2}\omega^1 - i_{X^1}\omega^2))$$

for uniquely determined by A real numbers a, b, c (or a, b, c with $c = 0$, respectively), i.e. roughly speaking, any such A coincides with the generalized Courant bracket up to three (or two, respectively) real constants.

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