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Generalized trend constants of Lipschitz mappings

Abstract. In 2015, Goebel and Bolibok defined the initial trend coefficient of a mapping and the class of initially nonexpansive mappings. They proved that the fixed point property for nonexpansive mappings implies the fixed point property for initially nonexpansive mappings. We generalize the above concepts and prove an analogous fixed point theorem. We also study the initial trend coefficient more deeply.

1. Introduction. Let X be a Banach space, C be a nonempty subset of X, and T be a mapping from C into itself. The mapping T is known as k-Lipschitz ($k \geq 0$) if

$$\|Tx - Ty\| \leq k \|x - y\|$$

for every $x, y \in C$. The minimal k, for which the above condition holds, is called the Lipschitz constant of T and is denoted by $k(T)$. If $k(T) < 1$ (resp. $k(T) \leq 1$), then T is said to be a contraction (resp. a nonexpansive mapping). By $\mathcal{L}_C(k)$ (or $\mathcal{L}(k)$ in short) we denote the set of all k-Lipschitz mappings from C into itself. The mapping T is Lipschitz if it is k-Lipschitz for some k.

Given vectors $u, v \in X$ and $x, y \in C$ such that $x \neq y$, we define functions $E_{u,v} : \mathbb{R} \to X$, $G_{u,v} : \mathbb{R} \to [0, \infty)$, $\varphi_{x,y} : \mathbb{R} \to [0, \infty)$ and $\psi_{x,y} : \mathbb{R} \to [0, \infty)$ by the formulas

$$E_{u,v}(t) = (1 - t) u + tv,$$

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$$G_{u,v}(t) = \|E_{u,v}(t)\| = \|(1-t)u + tv\|,$$
$$\varphi_{x,y}(t) = G_{x-y,\,Tx-Ty}(t) = \|(1-t)(x-y) + t(Tx-Ty)\|,$$

and

$$\psi_{x,y}(t) = \frac{\varphi_{x,y}(t)}{\|x-y\|} = \frac{\|(1-t)(x-y) + t(Tx-Ty)\|}{\|x-y\|}.$$

In [1] (see also [3]) the following coefficients were defined

$$\iota(T) = \sup\{\partial_+\psi_{x,y}(0) \,:\, x,\, y \in C,\, x \neq y\}, \tag{1.1}$$

and

$$\tau(T) = \sup\{\partial_-\psi_{x,y}(1) \,:\, x,\, y \in C,\, x \neq y\}, \tag{1.2}$$

where ∂_+ and ∂_- denote one-sided derivatives. The expression $\partial_+\psi_{x,y}(0)$ can be seen as the directional right derivative of the norm at the point $x-y$ along the vector $Tx - Ty - (x-y)$. The coefficients $\iota(T)$ and $\tau(T)$ are known as the initial and the final trend constants of T, respectively. The mapping T is said to be an initial contraction if $\iota(T) < 0$, and initially nonexpansive if $\iota(T) \leq 0$.

We extend the above notion in the following way. The trend constant (more precisely the right trend constant) of the mapping T at a point $\alpha \in \mathbb{R}$ is given by the formula

$$\iota_\alpha(T) = \sup\{\partial_+\psi_{x,y}(\alpha) \,:\, x,\, y \in C,\, x \neq y\}.$$

We say that $T : C \to C$ is a pre-initial contraction (resp. pre-initially nonexpansive) if it is a k-Lipschitz mapping, where $k > 1$, and there exists $\alpha \in \left(\frac{-1}{k-1}, 0\right]$ such that $\iota_\alpha(T) < 0$ (resp. $\iota_\alpha(T) \leq 0$). Note that the initial trend coefficient of T is equal to $\iota_0(T)$.

The mapping T is said to be firmly nonexpansive if for all $x, y \in C$ the function $\varphi_{x,y}(t)$ is nonincreasing on the interval $[0,1]$. The mapping T is firmly nonexpansive if and only if $\tau(T) \leq 0$.

The fixed point set for T is defined as

$$\mathrm{Fix}T = \{x \in C \,:\, Tx = x\}.$$

2. General trend of mappings. In this section, we generalize results obtained in [1]. Let $u,\, v \in X$. The function $t \mapsto G_{u,v}(t)$ is convex, so it is a semi-differentiable function at every real number t. Assume that $\alpha < \beta$. The following inequalities are obvious

$$\partial_- G_{u,v}(\alpha) \leq \partial_- G_{u,v}(\beta),$$
$$\partial_+ G_{u,v}(\alpha) \leq \partial_+ G_{u,v}(\beta),$$
$$\partial_- G_{u,v}(\alpha) \leq \partial_+ G_{u,v}(\alpha).$$

Claim 2.1. Let $u,\, v \in X$, $a,\, b,\, t_1 \in \mathbb{R}$, where $a < b$, and $t_2 = (1-t_1)a + t_1 b$. We have

$$\partial_- G_{(1-a)u+av,\,(1-b)u+bv}(t_1) = (b-a)\,\partial_- G_{u,v}(t_2). \tag{2.1}$$

Proof. Putting $w(s) = (1-s)u + sv$, where $s \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\partial_- G_{(1-a)u+av,\,(1-b)u+bv}(t_1) = \partial_- G_{w(a),\,w(b)}(t_1) \\
&= \lim_{h\to 0^-} \frac{\|(1-t_1-h)\,w(a) + (t_1+h)\,w(b)\| - \|(1-t_1)\,w(a) + t_1 w(b)\|}{h} \\
&= (b-a)\lim_{h\to 0^-} \frac{\|(1-t_2-h(b-a))u + (t_2+h(b-a))v\| - \|(1-t_2)u + t_2 v\|}{h\,(b-a)} \\
&= (b-a)\,\partial_- G_{u,v}(t_2).
\end{aligned}$$

□

Let C be a nonempty closed convex and bounded subset of X. Assume that $T : C \to C$ is k-Lipschitz for $k > 1$. Let $\alpha \in \left(\frac{-1}{k-1}, 0\right]$. Choose $A > k$ such that $\frac{-1}{A-1} < \alpha$. Given $x \in C$, consider the equation

$$y = \left(1 - \frac{1}{A}\right) x + \frac{1}{A} Ty. \tag{2.2}$$

Since the right hand side is a $\frac{k}{A}$-Lipschitz mapping and $\frac{k}{A} < 1$, this equation has a unique solution. Denoting this solution by Fx, we obtain the function $F : C \to C$ such that

$$Fx = \left(1 - \frac{1}{A}\right) x + \frac{1}{A} TFx. \tag{2.3}$$

Rearranging the above equality, we obtain

$$x = \frac{A}{A-1} Fx + \frac{-1}{A-1} TFx, \tag{2.4}$$

and

$$TFx = AFx - (A-1)\,x. \tag{2.5}$$

Since

$$\begin{aligned}
\|Fx - Fy\| &\le \left(1 - \frac{1}{A}\right) \|x - y\| + \frac{1}{A}\|TFx - TFy\| \\
&\le \left(1 - \frac{1}{A}\right) \|x - y\| + \frac{k}{A}\|Fx - Fy\|,
\end{aligned}$$

F is a $\frac{A-1}{A-k}$-Lipschitz mapping. For every $x \in C$ we define

$$F_\alpha x = (1-\alpha)\,Fx + \alpha TFx. \tag{2.6}$$

Observe that $F_\alpha x$ belongs to the line segment $[x, Fx]$, which is a subset of C, so F_α is a mapping from C into itself.

Let $x_0 \in C$. If $Fx_0 = x_0$, then from (2.3) we obtain $Tx_0 = x_0$. Conversely, if $Tx_0 = x_0$, then

$$x_0 = \left(1 - \frac{1}{A}\right) x_0 + \frac{1}{A} Tx_0.$$

This is an equality of the (2.2) form. We know that the equality (2.2) has a unique solution, so $Fx_0 = x_0$.

Assume that $F_\alpha x_0 = x_0$. It is equivalent to

$$(1-\alpha) Fx_0 + \alpha TFx_0 = x_0.$$

Applying the equality $TFx_0 = AFx_0 - (A-1) x_0$, we obtain

$$(1-\alpha) Fx_0 + \alpha (AFx_0 - (A-1) x_0) = x_0,$$

which is equivalent to $Fx_0 = x_0$. We have proved that $\mathrm{Fix}F_\alpha = \mathrm{Fix}F = \mathrm{Fix}T$.

Given distinct $x,\, y \in C$, we have

$$x = \frac{A}{A-1}Fx + \frac{-1}{A-1}TFx,$$

and

$$y = \frac{A}{A-1}Fy + \frac{-1}{A-1}TFy.$$

Observe that $Fx \neq Fy$. Putting $a = \frac{-1}{A-1}$, $b = \alpha$, $t_1 = 1$, $t_2 = \alpha$, $u = Fx - Fy$, and $v = TFx - TFy$ in the equality (2.1), we obtain

$$\begin{aligned}
&\frac{\partial_- G_{x-y,\, F_\alpha x - F_\alpha y}(1)}{\|x-y\|} \\
&\quad = \frac{\partial_- G_{(1-a)(Fx-Fy)+a(TFx-TFy),\,(1-b)(Fx-Fy)+b(TFx-TFy)}(1)}{\|x-y\|} \\
&\quad = \left(\alpha + \frac{1}{A-1}\right) \frac{\partial_- G_{Fx-Fy,\, TFx-TFy}(\alpha)}{\|x-y\|} \\
&\quad = \left(\alpha + \frac{1}{A-1}\right) \frac{\|Fx-Fy\|}{\|x-y\|} \cdot \frac{\partial_- G_{Fx-Fy,\, TFx-TFy}(\alpha)}{\|Fx-Fy\|} \\
&\quad \leq \left(\alpha + \frac{1}{A-1}\right) \frac{A-1}{A-k} \sup\left\{ \frac{\partial_- \varphi_{x,\,y}(\alpha)}{\|x-y\|} \,:\, x,\, y \in C,\, x \neq y \right\} \\
&\quad \leq \left(\alpha + \frac{1}{A-1}\right) \frac{A-1}{A-k} \sup\left\{ \frac{\partial_+ \varphi_{x,\,y}(\alpha)}{\|x-y\|} \,:\, x,\, y \in C,\, x \neq y \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\tau(F_\alpha) &= \sup\left\{ \frac{\partial_- G_{x-y,\, F_\alpha x - F_\alpha y}(1)}{\|x-y\|} \,:\, x,\, y \in C,\, x \neq y \right\} \\
&\leq \left(\alpha + \frac{1}{A-1}\right) \frac{A-1}{A-k} \sup\left\{ \frac{\partial_+ \varphi_{x,\,y}(\alpha)}{\|x-y\|} \,:\, x,\, y \in C,\, x \neq y \right\} \\
&= \left(\alpha + \frac{1}{A-1}\right) \frac{A-1}{A-k} \iota_\alpha(T).
\end{aligned}$$

As a special case, if $\alpha = 0$, we obtain

$$\begin{aligned}\tau(F) &= \sup\left\{\frac{\partial_- G_{x-y,\,F_0x-F_0y}(1)}{\|x-y\|} \,:\, x,\,y\in C,\,x\neq y\right\}\\ &\leq \frac{1}{A-k}\sup\left\{\frac{\partial_+\varphi_{x,\,y}(0)}{\|x-y\|} \,:\, x,\,y\in C,\,x\neq y\right\} = \frac{1}{A-k}\iota(T),\end{aligned}$$

and if $A = k+1$, then $\tau(F) \leq \iota(T)$. From the above consideration, we obtain the following corollaries.

Corollary 2.2. *If $T : C \to C$ is pre-initially nonexpansive, then there exists $\alpha \in \left(\frac{-1}{k-1}, 0\right]$ such that the mapping F_α given by* (2.6) *is firmly nonexpansive.*

We say that a subset $C \subset X$ has the fixed point property for nonexpansive mappings if every nonexpansive mapping $T : C \to C$ has a fixed point. The space X has the fixed point property for nonexpansive mappings if all nonempty closed convex and bounded subsets of X have this property. Similarly we define the fixed point properties for pre-initially nonexpansive mappings.

Corollary 2.3. *If a nonempty closed convex and bounded set $C \subset X$ has the fixed point property for nonexpansive mappings, then it has the fixed point property for pre-initially nonexpansive mappings.*

3. Formulas for trend constants. In this section, we provide a few formulas for trend constants of a mapping $T : C \to C$, where C is a nonempty subset of a Banach space X. Let us recall some basic facts about the subdifferential of the norm. Let $x, y \in X$. The function $\mathbb{R} \ni t \mapsto \|x+ty\|$ is convex, so the following limits exist

$$\begin{aligned}\partial_+\|x\|(y) &= \lim_{t\to 0^+}\frac{\|x+ty\|-\|x\|}{t},\\ \partial_-\|x\|(y) &= \lim_{t\to 0^-}\frac{\|x+ty\|-\|x\|}{t}.\end{aligned}$$

The subdifferential of the norm is defined by

$$\partial\|x\| = \{x^*\in X^* \,:\, \forall_{y\in X}\quad \partial_-\|x\|(y) \leq x^*(y) \leq \partial_+\|x\|(y)\}.$$

We have also the following formulas

$$\begin{aligned}\partial\|x\| &= \{x^*\in X^* \,:\, x^*(x) = \|x\|,\, \|x^*\| = 1\} \quad \text{if } x\neq 0,\\ \partial_+\|x\|(y) &= \max\{x^*(y) \,:\, x^*\in\partial\|x\|\},\\ \partial_-\|x\|(y) &= \min\{x^*(y) \,:\, x^*\in\partial\|x\|\}.\end{aligned}$$

Claim 3.1. Let $u : [a,b] \to X$ be a differentiable function at $t_0 \in [a,b]$. Then the function $\gamma(t) = \|u(t)\|$, $t \in [a,b]$ is differentiable on the left at $t_0 \in (a,b]$

and on the right at $t_0 \in [a,b)$. Moreover, we have the following chain rules

$$\partial_+\gamma(t_0) = \partial_+ \|u(t_0)\| \left(u'(t_0)\right) = \max\{x^*\left(u'(t_0)\right) : x^* \in \partial \|u(t_0)\|\}, \tag{3.1}$$

$$\partial_-\gamma(t_0) = \partial_- \|u(t_0)\| \left(u'(t_0)\right) = \min\left\{x^*\left(u'\left(t_0\right)\right) : x^* \in \partial \|u\left(t_0\right)\|\right\}. \tag{3.2}$$

For further details about subdifferentials of norms, see for example [2].

Let $u(t) = (1-t)(x-y) + t(Tx - Ty)$ and $\alpha \in \mathbb{R}$. Then, by (3.1),

$$\begin{aligned}\partial_+\varphi_{x,y}(\alpha) &= \max\left\{x^*\left(u'(\alpha)\right) : x^* \in \partial \|u(\alpha)\|\right\} \\ &= \max\left\{x^*\left(Tx - Ty - (x-y)\right) : x^* \in \partial \|u(\alpha)\|\right\}\end{aligned}$$

for every $\alpha \in \mathbb{R}$. As a consequence, we obtain a new formula for the trend constant of T at α.

Corollary 3.2. *For every $\alpha \in \mathbb{R}$,*

$$\iota_\alpha(T) = \sup\left\{\frac{x^*\left(Tx - Ty - (x-y)\right)}{\|x-y\|}\right\},$$

where the supremum is taken over all distinct vectors $x, y \in C$ and functionals $x^* \in \partial \|(1-\alpha)(x-y) + \alpha(Tx - Ty)\|$. In case of the initial trend coefficient, this formula takes the following form

$$\iota(T) = \sup\left\{\frac{x^*(Tx-Ty)}{\|x-y\|} - 1 : x, y \in C,\, x \neq y,\, x^* \in \partial\|x-y\|\right\}. \tag{3.3}$$

Theorem 3.3. *Let $x, y \in C$ be distinct vectors, and $\alpha \in \mathbb{R}$. If*

$$E_t^* \in \partial \|E_{x-y,\,Tx-Ty}(t)\|$$

for every $t > \alpha$, then

$$\partial_+\varphi_{x,y}(\alpha) = \lim_{t\to\alpha^+} E_t^*\left(Tx - Ty - (x-y)\right). \tag{3.4}$$

Proof. Let $t_0 > t_1 > t_2 = \alpha$, and let $E(t) = E_{x-y,\,Tx-Ty}(t)$ for $t \geq \alpha$. We obtain

$$\begin{aligned}\frac{\|E(t_1)\| - \|E(t_2)\|}{t_1 - t_2} &\leq \frac{\|E(t_1)\| - \left\|E_{t_1}^*\right\| \|E(t_2)\|}{t_1 - t_2} \\ &\leq \frac{E_{t_1}^*\left(E(t_1)\right) - E_{t_1}^*\left(E(t_2)\right)}{t_1 - t_2} \leq E_{t_1}^*\left(\frac{E(t_1) - E(t_2)}{t_1 - t_2}\right) \\ &= E_{t_1}^*\left(Tx - Ty - (x-y)\right)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}E_{t_1}^*\left(Tx - Ty - (x-y)\right) &= E_{t_1}^*\left(\frac{E(t_0) - E(t_1)}{t_0 - t_1}\right) \\ &= \frac{E_{t_1}^*\left(E(t_0)\right) - E_{t_1}^*\left(E(t_1)\right)}{t_0 - t_1} \\ &\leq \frac{\|E(t_0)\| - \|E(t_1)\|}{t_0 - t_1}.\end{aligned} \tag{3.6}$$

The equality (3.4) follows from the above inequalities and the following equalities

$$\lim_{t_1\to\alpha^+}\frac{\|E(t_1)\|-\|E(t_2)\|}{t_1-t_2}=\lim_{t_0\to\alpha^+}\lim_{t_1\to\alpha^+}\frac{\|E(t_0)\|-\|E(t_1)\|}{t_0-t_1}=\partial_+\varphi_{x,y}(\alpha).$$

□

From the above theorem we obtain another formula for the trend constant of the mapping T at α.

Corollary 3.4. *If* $E_t^*\in\partial\|E_{x-y,Tx-Ty}(t)\|$ for $t>\alpha$, *then*

$$\iota_\alpha(T)=\sup\left\{\frac{\lim_{t\to\alpha^+}E_t^*(Tx-Ty-(x-y))}{\|x-y\|}\ :\ x,\,y\in C,\,x\neq y\right\}.$$

In [1] the formula for the initial trend coefficient for Hilbert spaces is given. Using the equality (3.3), we can calculate formulas for this coefficient in some spaces. Here we will deal with the space $C[a,b]$. In order to prove such a result, we can use the characterization of the subdifferential of the norm in $C[a,b]$ given in [2]. Note that in the literature, one can find similar characterizations for some other spaces (see for example [2] and [4]). Another approach is to apply a formula for the directional right derivative of the norm. Such a formula for $C[a,b]$ is given in [5]. Using one of the above methods, we obtain the following claim.

Claim 3.5. Let C be a nonempty subset of the space $C[a,b]$, and let $M_0(z)=\{t\in[a,b]\ :\ |z(t)|=\|z\|\}$, $z\in C[a,b]$. Given a mapping $T:C\to C$,

$$\iota(T)=\sup\left\{\frac{((Tx)(s)-(Ty)(s))\operatorname{sgn}(x(s)-y(s))}{\|x-y\|}-1\right\},\tag{3.7}$$

where the supremum is taken over all distinct vectors $x,\,y\in C$ and all s in $M_0(x-y)$.

Example 3.6. In the two-dimensional Euclidean plane $\mathbb{R}^2$ we consider the k-Lipschitz mapping

$$T_{k,\beta}(x,y)=k(x\cos\beta-y\sin\beta,x\sin\beta+y\cos\beta),$$

where $k>1$, $\beta\in\mathbb{R}$. One can easily compute the initial trend constant of $T_{k,\beta}$ and the trend constant at $\alpha>\frac{-1}{k-1}$:

$$\iota(T_{k,\beta}(x,y))=k\cos\beta-1,$$

$$\iota_\alpha(T_{k,\beta}(x,y))=\frac{\alpha-1+(1-2\alpha)k\cos\beta+\alpha k^2}{\sqrt{(1-\alpha)^2+2(1-\alpha)\alpha k\cos\beta+\alpha^2k^2}}.$$

Taking $k=5$, $\beta=\arccos\frac{4}{5}$, and $\alpha=-\frac{1}{6}$, we obtain $\iota(T_{k,\beta})=3$, and $\iota_\alpha(T_{k,\beta})=0$. Therefore, in this case $T_{k,\beta}(x,y)$ is pre-initially nonexpansive but isn't initially nonexpansive.

4. Remarks about the initial trend coefficient. We say that the norm $\|\cdot\|$ of a Banach space X is Gâteaux differentiable at a point $x \in X$ if for every $h \in X$ the limit

$$\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t}$$

exists. If, moreover, this limit is uniform for $x,\, h \in S_X$, then we say that the norm is uniformly Fréchet differentiable. We say that the norm is Gâteaux differentiable if it is Gâteaux differentiable at every point $x \in S_X$.

Theorem 4.1. *Let X be a Banach space whose norm is uniformly Fréchet differentiable and C be a nonempty and convex subset of X. If $T : C \to C$ is a Lipschitz mapping such that $\iota(T) < 0$, then there exists $\delta \in (0,\, 1)$ such that the mapping $(1 - t_0)\, I + t_0 T$ is a contraction, where I is the identity on C and $t_0 \in (0,\, \delta)$.*

Proof. Assume that $T \in \mathcal{L}(k)$. Let $\varepsilon = \frac{|\iota(T)|}{2(k+1)}$. Since the norm of X is uniformly Fréchet differentiable, there exists $\tau > 0$ such that

$$\frac{\|z + th\| - \|z\|}{t} < \varepsilon + z^*(h) \tag{4.1}$$

for every $t \in (0,\, \tau)$ and $z,\, h \in S_X$, where $z^* \in \partial \|z\|$. Note that $\partial \|z\|$ is a one-element set, because the norm of X is Gâteaux differentiable. We choose $t_0 \in \left(0,\, \frac{\tau}{1+k}\right)$, and define the mapping $T_{t_0} : C \to C$ by the formula $T_{t_0} x = (1 - t_0)\, x + t_0 T x$.

Given distinct elements $x,\, y \in C$, we have $x - y \neq Tx - Ty$. Otherwise, $\psi_{x,\,y}(t) = 1$ for $t \in [0, 1]$, and $\iota(T) \geq \partial_+ \psi_{x,\,y}(0) = 0$, which contradicts our assumption. Put $u = x - y$, and $v = Tx - Ty$. Note that

$$\frac{t_0 \|v - u\|}{\|u\|} \leq \frac{t_0 (\|Tx - Ty\| + \|x - y\|)}{\|x - y\|} \leq t_0 (1 + k) < \tau.$$

Using this inequality and putting $z = \frac{u}{\|u\|}$, $h = \frac{v-u}{\|v-u\|}$, $t = \frac{t_0\|v-u\|}{\|u\|}$ in (4.1), we obtain

$$\begin{aligned}
\|T_{t_0} x - T_{t_0} y\| &= \|u + t_0 (v - u)\| \\
&= t_0 \|v - u\| \frac{\left\| \frac{u}{\|u\|} + \frac{t_0\|v-u\|}{\|u\|} \cdot \frac{v-u}{\|v-u\|} \right\| - \left\| \frac{u}{\|u\|} \right\|}{\frac{t_0\|v-u\|}{\|u\|}} + \|u\| \\
&< t_0 \|v - u\| \left(\varepsilon + z^* \left(\frac{v - u}{\|v - u\|} \right) \right) + \|u\| \\
&\leq \left(t_0 \left(\frac{|\iota(T)|}{2} + z^* \left(\frac{v - u}{\|u\|} \right) \right) + 1 \right) \|u\| \\
&= \left(t_0 \left(\frac{|\iota(T)|}{2} + z^* \left(\frac{Tx - Ty - (x - y)}{\|x - y\|} \right) \right) + 1 \right) \|x - y\|,
\end{aligned}$$

where $z^* \in \partial \left\| \frac{u}{\|u\|} \right\|$. Since $\partial \left\| \frac{u}{\|u\|} \right\| = \partial \|x - y\|$, applying Corollary 3.2, we get

$$\begin{aligned} \|T_{t_0}x - T_{t_0}y\| &< \left(t_0 \left(\frac{|\iota(T)|}{2} + \iota(T) \right) + 1 \right) \|x - y\| \\ &= \left(1 + \frac{\iota(T)\, t_0}{2} \right) \|x - y\| . \end{aligned}$$

Since $1 + \frac{\iota(T) t_0}{2} < 1$, the mapping T_{t_0} is a contraction. □

Let C be a nonempty subset of a Banach space X. Consider a Lipschitz mapping $T : C \to C$. Let $\mu(t) = \sup\{\psi_{x,y}(t) \,:\, x, y \in C, x \neq y\}$, $t \in \mathbb{R}$. The function μ is convex. We define the coefficient

$$\kappa(T) = \partial_+ \mu(0),$$

which is similar to the initial trend coefficient (1.1), but the derivative and the supremum are swapped. This coefficient is greater than or equal to the initial trend coefficient of T. Indeed, given $\varepsilon > 0$, there exist distinct elements $x, y \in C$ such that

$$\iota(T) \leq \lim_{t \to 0^+} \frac{\psi_{x,y}(t) - 1}{t} + \varepsilon,$$

and therefore

$$\begin{aligned} \iota(T) &\leq \lim_{t \to 0^+} \frac{\sup\{\psi_{x,y}(t) \,:\, x, y \in C, x \neq y\} - 1}{t} + \varepsilon \\ &= \lim_{t \to 0^+} \frac{\mu(t) - \mu(0)}{t} + \varepsilon \\ &= \kappa(T) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\iota(T) \leq \kappa(T)$. In the linear case we have the equality.

Theorem 4.2. *For any linear bounded mapping $T : X \to X$, $\iota(T) = \kappa(T)$.*

Proof. Observe that, by the assumptions of the theorem,

$$\mu(t) = \sup\{\|(1 - t)x + tTx\| \,:\, x \in S_X\},$$

and according to Corollary 3.2,

$$\iota(T) = \sup\{x^*(Tx) - 1 \,:\, x \in S_X, x^* \in \partial \|x\|\}.$$

In case of $T = 0$, we have $\iota(T) = \kappa(T) = -1$. Now we can assume that $T \neq 0$. Let $\varepsilon > 0$ and $t_0 \in \left(0, \min\left\{\frac{\varepsilon}{4\|T\|(1+\|T\|)}, \frac{1}{1+\|T\|}\right\}\right)$. There exists an element $x_0 \in S_X$ such that

$$\kappa(T) \leq \frac{\mu(t_0) - \mu(0)}{t_0} \leq \frac{\|(1 - t_0)x_0 + t_0 T x_0\| + \frac{\varepsilon t_0}{2} - 1}{t_0}. \tag{4.2}$$

Let $u = (1 - t_0) x_0 + t_0 T x_0$ and $u^* \in \partial \|u\|$. Then

$$\|u\| \geq \|(1 - t_0) x_0\| - \|t_0 T x_0\| \geq 1 - t_0 (1 + \|T\|) > 0.$$

We obtain

$$\begin{aligned}
\kappa(T) &\leq \frac{u^* ((1 - t_0) x_0 + t_0 T x_0) - 1}{t_0} + \frac{\varepsilon}{2} \\
&= \frac{(1 - t_0) u^* (x_0) + t_0 u^* (T x_0) - 1}{t_0} + \frac{\varepsilon}{2} \\
&\leq u^* (T x_0) - 1 + \frac{\varepsilon}{2} \\
&= u^* \left(\frac{Tu}{\|u\|} \right) - 1 + u^* \left(Tu - \frac{Tu}{\|u\|} \right) + u^* (T x_0 - Tu) + \frac{\varepsilon}{2} \\
&\leq u^* \left(T \left(\frac{u}{\|u\|} \right) \right) - 1 + \|T\| \left\| u - \frac{u}{\|u\|} \right\| + \|T\| \|t_0 (x_0 - T x_0)\| + \frac{\varepsilon}{2} \\
&\leq \iota(T) + \|T\| \left| \|u\| - 1 \right| + t_0 \|T\| (1 + \|T\|) + \frac{\varepsilon}{2} \\
&\leq \iota(T) + \|T\| \|u - x_0\| + t_0 \|T\| (1 + \|T\|) + \frac{\varepsilon}{2} \\
&\leq \iota(T) + 2 t_0 \|T\| (1 + \|T\|) + \frac{\varepsilon}{2} \\
&\leq \iota(T) + \varepsilon.
\end{aligned}$$

Since ε is an arbitrary positive number, $\kappa(T) \leq \iota(T)$. □

The next claim gives us the characterization of mappings such that $\kappa(T) < 0$. Since for a linear bounded self-mappings T we have $\iota(T) = \kappa(T)$, this claim also applies to the linear initial contractions.

Claim 4.3. Let $T : C \to C$ be such that $\kappa(T) < 0$. Then there exist a contraction $R : C \to C$ and $\delta > 0$ such that $Tx = \frac{1}{\delta} Rx + \left(1 - \frac{1}{\delta}\right) x$ for every $x \in C$.

Proof. There exists $\delta > 0$ such that

$$\left| \frac{\mu(t) - 1}{t} - \kappa(T) \right| \leq \frac{|\kappa(T)|}{2}$$

for $t \in (0, \delta]$. Therefore,

$$\frac{\mu(t) - 1}{t} \leq \frac{\kappa(T)}{2}.$$

For every $x, y \in C$, $x \neq y$ we get

$$\frac{\|(1 - \delta)(x - y) + \delta (Tx - Ty)\|}{\|x - y\|} \leq 1 + \frac{\kappa(T) \delta}{2}.$$

Putting $Rx = (1 - \delta) x + \delta T x$ for $x \in C$, we obtain

$$\|Rx - Ry\| \leq \left(1 + \frac{\kappa(T) \delta}{2}\right) \|x - y\|,$$

where $1 + \frac{\kappa(T)\delta}{2} < 1$, so R is a contraction. □

Let C be a subset of the space $X = C[0,1]$. We define the Hammerstein operator $T : C \to X$ by the following formula

$$(Tu)(s) = \int_0^1 k(s, t) f(t, u(t))\, dt, \quad s \in [0, 1],$$

where k, f are continuous functions on $[0,1]^2$, $t \to f(t_0, t)$ is a $k_f(t_0)$-Lipschitz mapping on the interval $[0,1]$, and the function $k_f : [0,1] \to [0,\infty)$ is Lebesgue integrable.

Claim 4.4. If $C = B_X$ or $C = X$, then for the above mapping T, $\iota(T) = k(T) - 1$.

Proof. The inequality $\iota(T) \le k(T) - 1$ is true in general (see [1]), so it is enough to prove the opposite inequality.

Let $U > \max\{|k(s, t)| : s, t \in [0,1]\}$. Given $\varepsilon > 0$, there exist distinct elements $x, y \in C$ such that

$$\frac{\|Tx - Ty\|}{\|x - y\|} \ge k(T) - \frac{\varepsilon}{2}.$$

For $z \in X$ we put $M_0(z) = \{t \in [a,b] : |z(t)| = \|z\|\}$. We choose an arbitrary $s_0 \in M_0(x - y)$ and $s \in [0,1]$. By the absolute continuity of the Lebesgue integral there exists $\delta > 0$ such that $\int_A k_f(t)\, d\mu(t) \le \frac{\varepsilon}{4U}$ if $\mu(A) \le 2\delta$. For each $h \in C[0,1]$ and $v \in \mathbb{R}$ we define $F_{h,v} \in C[0,1]$ as follows

$$F_{h,v}(t) = \begin{cases} h(t), & t \in [0, 1] \setminus (s - \delta, s + \delta), \\ \frac{v - h(s-\delta)}{\delta}(t - s) + v, & t \in (s - \delta, s] \cap [0, 1], \\ \frac{h(s+\delta) - v}{\delta}(t - s) + v, & t \in [s, s + \delta) \cap [0, 1]. \end{cases}$$

For $i \in \{0,1\}$ we define functions $x_i, y_i \in C[0,1]$ by the formulas

$$x_0(t) = F_{x, x(s_0)}(t),$$

$$y_0(t) = F_{y, y(s_0)}(t),$$

$$x_1(t) = F_{x, y(s_0)}(t),$$

$$y_1(t) = F_{y, x(s_0)}(t).$$

Setting $L = \max\{0, s-\delta\}$ and $R = \min\{1, s+\delta\}$, we obtain

$$
\begin{aligned}
&|((Tx_i - Ty_i) - (Tx - Ty))(s)| \\
&= \left| \int_L^R k(s,t)\left((f(t, x_i(t)) - f(t, x(t))) - (f(t, y_i(t)) - f(t, y(t)))\right) dt \right| \\
&\le \int_L^R |k(s,t)| \left(|f(t, x_i(t)) - f(t, y_i(t))| + |f(t, x(t)) - f(t, y(t))|\right) dt \\
&\le \int_{[L,R]} Uk_f(t)\left(|x_i(t) - y_i(t)| + |x(t) - y(t)|\right) d\mu(t) \\
&\le \int_{[L,R]} Uk_f(t)\left(\|x_i - y_i\| + \|x - y\|\right) d\mu(t) \\
&= 2(\|x-y\|)\, U \int_{[L,R]} k_f(t)\, d\mu(t) \\
&\le \frac{\varepsilon \|x-y\|}{2}.
\end{aligned}
$$

Since $s \in M_0(x_i - y_i)$ for $i \in \{0,1\}$, by the equality (3.3),

$$
\begin{aligned}
\iota(T) &\ge \left(\frac{(Tx_i)(s) - (Ty_i)(s)}{\|x_i - y_i\|}\right) \operatorname{sgn}(x_i(s) - y_i(s)) - 1 \\
&= \left(\frac{(Tx_i)(s) - (Ty_i)(s)}{\|x - y\|}\right) (-1)^i \operatorname{sgn}(x(s_0) - y(s_0)) - 1 \\
&\ge \left(\frac{(Tx)(s) - (Ty)(s)}{\|x - y\|}\right) (-1)^i \operatorname{sgn}(x(s_0) - y(s_0)) - \frac{\varepsilon}{2} - 1.
\end{aligned}
$$

Because $i \in \{0,1\}$ is arbitrary, we obtain

$$\iota(T) \ge \frac{|(Tx)(s) - (Ty)(s)|}{\|x-y\|} - \frac{\varepsilon}{2} - 1.$$

The number $s \in [0,1]$ is also arbitrary, thus

$$\iota(T) \ge \frac{\|Tx - Ty\|}{\|x-y\|} - \frac{\varepsilon}{2} - 1 \ge k(T) - \varepsilon - 1,$$

and finally we apply the fact that $\varepsilon > 0$ is arbitrary, therefore $\iota(T) \ge k(T) - 1$. □

At the end, we will study some examples from [1].

Example 4.5. Let $C = [a, b]$ and $f : C \to C$, $f \in \mathcal{L}(k)$, $k > 1$. It is known that

$$\iota(f) = \sup\left\{\frac{f(x) - f(y)}{x - y} : x, y \in [a, b], x \neq y\right\} - 1.$$

The Lipschitz constant of f is given by

$$k(f) = \sup\left\{\left|\frac{f(x) - f(y)}{x - y}\right| : x, y \in [a, b], x \neq y\right\},$$

so the initial trend coefficient of f can be smaller than $k(f) - 1$. Let $\alpha \in (0, 1)$, $b \geq x \geq y \geq a$ and $g(t) = \frac{2-\alpha}{k+1} f(t) + \frac{k+\alpha-1}{k+1} t$, $t \in [a, b]$. Note that $g : C \to C$. We have

$$\begin{aligned} g(x) - g(y) &= \frac{2-\alpha}{k+1}(f(x) - f(y)) + \frac{k+\alpha-1}{k+1}(x - y) \\ &\leq \frac{2-\alpha}{k+1}(\iota(f) + 1)(x - y) + \frac{k+\alpha-1}{k+1}(x - y) \\ &= \left(1 + \frac{2-\alpha}{k+1}\iota(f)\right)(x - y) \end{aligned}$$

and

$$\begin{aligned} g(x) - g(y) &\geq \frac{2-\alpha}{k+1}(-k)(x - y) + \frac{k+\alpha-1}{k+1}(x - y) \\ &= (\alpha - 1)(x - y). \end{aligned}$$

Therefore, $g \in \mathcal{L}\big(\max\big\{1 + \frac{2-\alpha}{k+1}\iota(f), 1 - \alpha\big\}\big)$. Thus f is the following affine combination of g and the identity:

$$f(t) = \frac{k+1}{2-\alpha} g(t) - \frac{k+\alpha-1}{2-\alpha} t$$

and g is a nonexpansive mapping (resp. a contraction) provided that $\iota(f) \leq 0$ (resp. $\iota(f) < 0$).

Example 4.6. Let $X = C[a, b]$, $r > 0$ and $B(0, r) = \{x \in X : \|x\| \leq r\}$. Assume that the function $f : [-r, r] \to [-r, r]$ is of class $\mathcal{L}(k)$. It is known that the composition operator $F : B(0, r) \to B(0, r)$ defined by

$$Fx(t) = f(x(t))$$

has the initial trend coefficient given by

$$\iota(F) = \sup\left\{\frac{f(x) - f(y)}{x - y} : x, y \in [a, b], x \neq y\right\} - 1.$$

We define α and g as in the previous example. Consider the composition operator $G : B(0, r) \to B(0, r)$ defined by

$$Gx(t) = g(x(t)) = \frac{2-\alpha}{k+1} f(x(t)) + \frac{k+\alpha-1}{k+1} x(t).$$

Let $x, y \in B(0, r)$, $s \in [a, b]$. By symmetry, we can assume that $x(s) \geq y(s)$. We obtain

$$\begin{aligned} Gx(s) - Gy(s) &= \frac{2-\alpha}{k+1}\left(f(x(s)) - f(y(s))\right) + \frac{k+\alpha-1}{k+1}\left(x(s) - y(s)\right) \\ &\leq \frac{2-\alpha}{k+1}(\iota(F)+1)(x(s)-y(s)) + \frac{k+\alpha-1}{k+1}(x(s)-y(s)) \\ &\leq \left(1 + \frac{2-\alpha}{k+1}\iota(f)\right)\|x-y\| \end{aligned}$$

and

$$\begin{aligned} Gx(s) - Gy(s) &\geq \frac{2-\alpha}{k+1}(-k)\left(x(s) - y(s)\right) + \frac{k+\alpha-1}{k+1}\left(x(s) - y(s)\right) \\ &= (\alpha - 1)\left(x(s) - y(s)\right) \\ &\geq (\alpha - 1)\|x - y\|. \end{aligned}$$

Thus, $G \in \mathcal{L}\big(\max\big\{1+\frac{2-\alpha}{k+1}\iota(F), 1-\alpha\big\}\big)$ and F is an affine combination of G and the identity, where G is a nonexpansive mapping (resp. a contraction) provided that $\iota(F) \leq 0$ (resp. $\iota(F) < 0$).

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