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Cobalancing hybrid numbers

Abstract. Hybrid numbers are generalization of complex, hyperbolic and dual numbers. In this paper, we define and study hybrid numbers with cobalancing and Lucas-cobalancing coefficients. We derive some fundamental identities for these numbers, among others the Binet formulas and the general bilinear index-reduction formulas which imply the Catalan, Cassini, Vajda, d'Ocagne and Halton identities. Moreover, the generating functions for cobalancing and Lucas-cobalancing hybrid numbers are presented.

1. Introduction. A positive integer n is a balancing number if it is the solution of the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1}$$

for some positive integer r. Here r is called the balancer corresponding to the balancing number n. The sequence of balancing numbers, denoted by $\{B_n\}$, was introduced by Behera and Panda in [1]. It is well known that n is a balancing number if and only if n^2 is a triangular number, i.e. $8n^2+1$ is a perfect square, see [1].

In [6], Panda introduced the sequence of Lucas-balancing numbers, denoted by $\{C_n\}$ and defined as follows: if B_n is a balancing number, the number C_n for which $(C_n)^2=8B_n^2+1$ is called a Lucas-balancing number.

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Cobalancing numbers were defined and introduced in [7] by modification of formula (1). The authors called a positive integer n a cobalancing number with cobalancer r if

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r).$$

Let b_n denote the nth cobalancing number. The nth Lucas-cobalancing number c_n is defined with $(c_n)^2 = 8b_n^2 + 8b_n + 1$, see [4, 5].

The balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers fulfill the following recurrence relations

$$B_n = 6B_{n-1} - B_{n-2} \text{ for } n \geq 2, \text{ with } B_0 = 0, \ B_1 = 1,$$

$$C_n = 6C_{n-1} - C_{n-2} \text{ for } n \geq 2, \text{ with } C_0 = 1, \ C_1 = 3,$$

$$b_n = 6b_{n-1} - b_{n-2} + 2 \text{ for } n \geq 2, \text{ with } b_0 = 0, \ b_1 = 0, \tag{2}$$

$$c_n = 6c_{n-1} - c_{n-2} \text{ for } n \geq 2, \text{ with } c_0 = -1, \ c_1 = 1. \tag{3}$$

Note that cobalancing and Lucas-cobalancing numbers were originally defined for $n \geq 1$. Defining $b_0 = 0$ and $c_0 = -1$, we obtain the same, correctly defined sequences.

The Binet type formulas for the above-mentioned sequences have the following forms

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

$$C_n = \frac{\alpha^n + \beta^n}{2},$$

$$b_n = \frac{\alpha^{n-\frac{1}{2}} - \beta^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2}, \tag{4}$$

$$c_n = \frac{\alpha^{n-\frac{1}{2}} + \beta^{n-\frac{1}{2}}}{2}, \tag{5}$$

for $n \geq 0$, where

$$\alpha = 3 + 2\sqrt{2}, \ \beta = 3 - 2\sqrt{2}, \ \alpha^{\frac{1}{2}} = 1 + \sqrt{2}, \ \beta^{\frac{1}{2}} = 1 - \sqrt{2}. \tag{6}$$

Note that additionally defined b_0 and c_0 satisfy (4)–(5).

Table 1 includes initial terms of the balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers for $0 \leq n \leq 7$.

n	0	1	2	3	4	5	6	7
B_n	0	1	6	35	204	1189	6930	40391
C_n	1	3	17	99	577	3363	19601	114243
b_n	0	0	2	14	84	492	2870	16730
c_n	-1	1	7	41	239	1393	8119	47321

Table 1. The balancing type numbers.

In [3], Özdemir introduced a new non-commutative number system called hybrid numbers. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined by

$$\mathbb{K} = \{\mathbf{z} = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} \colon a, b, c, d \in \mathbb{R}\},$$

where

$$\mathbf{i}^2 = -1,\ \boldsymbol{\varepsilon}^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}. \tag{7}$$

Two hybrid numbers $\mathbf{z_1} = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}$, $\mathbf{z_2} = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}$ are equal if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$. The sum and subtraction of two hybrid numbers are defined by $\mathbf{z_1} \pm \mathbf{z_2} = (a_1 \pm a_2) + (b_1 \pm b_2)\mathbf{i} + (c_1 \pm c_2)\boldsymbol{\varepsilon} + (d_1 \pm d_2)\mathbf{h}$. The addition operation is commutative and associative, zero is the null element. With respect to the addition operation, the inverse element of $\mathbf{z} = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$ is $-\mathbf{z} = -a - b\mathbf{i} - c\boldsymbol{\varepsilon} - d\mathbf{h}$. Hence $(\mathbb{K}, +)$ is an Abelian group.

Using (7), we get the multiplication Table 2.

$\cdot$	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\boldsymbol{\varepsilon} + \mathbf{i}$
$\boldsymbol{\varepsilon}$	$1 + \mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$-(\boldsymbol{\varepsilon} + \mathbf{i})$	$\boldsymbol{\varepsilon}$	1

TABLE 2. The hybrid number multiplication.

In the literature, we can find many papers in which hybrid numbers with coefficients being consecutive terms of known sequences are investigated, see for example [2, 9, 10].

The balancing and Lucas-balancing hybrid numbers were introduced in [2] as follows. For a nonnegative integer n the nth balancing hybrid number BH_n is defined as

$$BH_n = B_n + B_{n+1}\mathbf{i} + B_{n+2}\boldsymbol{\varepsilon} + B_{n+3}\mathbf{h},$$

where B_n denotes the nth balancing number. The nth Lucas-balancing hybrid number CH_n was defined as

$$CH_n = C_n + C_{n+1}\mathbf{i} + C_{n+2}\boldsymbol{\varepsilon} + C_{n+3}\mathbf{h},$$

where C_n is nth Lucas-balancing number.

In this paper, we define and study cobalancing and Lucas-cobalancing hybrid sequences.

2. Main results. Let $n \geq 0$ be an integer. The nth cobalancing hybrid number bH_n and the nth Lucas-cobalancing hybrid number cH_n are defined as

$$bH_n = b_n + b_{n+1}\mathbf{i} + b_{n+2}\boldsymbol{\varepsilon} + b_{n+3}\mathbf{h} \tag{8}$$

and

$$cH_n = c_n + c_{n+1}\mathbf{i} + c_{n+2}\boldsymbol{\varepsilon} + c_{n+3}\mathbf{h}, \tag{9}$$

respectively, where b_n is the nth cobalancing number, c_n is the nth Lucas-cobalancing number and $\mathbf{i}$, $\boldsymbol{\varepsilon}$, $\mathbf{h}$ are hybrid units which satisfy (7).

Using (8), (9) and Table 1, we get

$$\begin{aligned}
bH_0 &= 2\boldsymbol{\varepsilon} + 14\mathbf{h},\\
bH_1 &= 2\mathbf{i} + 14\boldsymbol{\varepsilon} + 84\mathbf{h},\\
cH_0 &= -1 + \mathbf{i} + 7\boldsymbol{\varepsilon} + 41\mathbf{h},\\
cH_1 &= 1 + 7\mathbf{i} + 41\boldsymbol{\varepsilon} + 239\mathbf{h}.
\end{aligned}$$

Theorem 1. *Let $n \geq 2$ be an integer. Then*

$$bH_n = 6bH_{n-1} - bH_{n-2} + 2\left(1 + \mathbf{i} + \boldsymbol{\varepsilon} + \mathbf{h}\right),$$

where $bH_0 = 2\boldsymbol{\varepsilon} + 14\mathbf{h}$, $bH_1 = 2\mathbf{i} + 14\boldsymbol{\varepsilon} + 84\mathbf{h}$.

Proof. By formulas (8) and (2), we get

$$\begin{aligned}
&6bH_{n-1} - bH_{n-2}\\
&= 6\left(b_{n-1} + b_n\mathbf{i} + b_{n+1}\boldsymbol{\varepsilon} + b_{n+2}\mathbf{h}\right) - \left(b_{n-2} + b_{n-1}\mathbf{i} + b_n\boldsymbol{\varepsilon} + b_{n+1}\mathbf{h}\right)\\
&= \left(6b_{n-1} - b_{n-2}\right) + \left(6b_n - b_{n-1}\right)\mathbf{i} + \left(6b_{n+1} - b_n\right)\boldsymbol{\varepsilon} + \left(6b_{n+2} - b_{n+1}\right)\mathbf{h}\\
&= (b_n - 2) + (b_{n+1} - 2)\mathbf{i} + (b_{n+2} - 2)\boldsymbol{\varepsilon} + (b_{n+3} - 2)\mathbf{h}\\
&= bH_n - \left(2 + 2\mathbf{i} + 2\boldsymbol{\varepsilon} + 2\mathbf{h}\right),
\end{aligned}$$

which ends the proof. □

In the same way, using (9) and (3), one can easily prove the next theorem.

Theorem 2. *Let $n \geq 2$ be an integer. Then*

$$cH_n = 6cH_{n-1} - cH_{n-2},$$

where $cH_0 = -1 + \mathbf{i} + 7\boldsymbol{\varepsilon} + 41\mathbf{h}$, $cH_1 = 1 + 7\mathbf{i} + 41\boldsymbol{\varepsilon} + 239\mathbf{h}$.

Hence we get the Binet formulas for the cobalancing and Lucas-cobalancing hybrid numbers.

Theorem 3. *Let $n \geq 0$ be an integer. Then*

$$bH_n = \frac{\alpha^{n-\frac{1}{2}}}{\alpha - \beta}\hat{\alpha} - \frac{\beta^{n-\frac{1}{2}}}{\alpha - \beta}\hat{\beta} - \frac{1}{2}\hat{\mathbf{1}}, \tag{10}$$

where α, β are given by (6) and

$$\hat{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\varepsilon} + \alpha^3\mathbf{h},\ \hat{\beta} = 1 + \beta\mathbf{i} + \beta^2\boldsymbol{\varepsilon} + \beta^3\mathbf{h},\ \hat{\mathbf{1}} = 1 + \mathbf{i} + \boldsymbol{\varepsilon} + \mathbf{h}. \tag{11}$$

Proof. Using (8) and (4), we have

$$\begin{aligned}
bH_n &= b_n + b_{n+1}\mathbf{i} + b_{n+2}\boldsymbol{\varepsilon} + b_{n+3}\mathbf{h}\\
&= \left(\frac{\alpha^{n-\frac{1}{2}} - \beta^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{1}{2}\right) + \left(\frac{\alpha^{n+1-\frac{1}{2}} - \beta^{n+1-\frac{1}{2}}}{\alpha-\beta} - \frac{1}{2}\right)\mathbf{i}\\
&+ \left(\frac{\alpha^{n+2-\frac{1}{2}} - \beta^{n+2-\frac{1}{2}}}{\alpha-\beta} - \frac{1}{2}\right)\boldsymbol{\varepsilon} + \left(\frac{\alpha^{n+3-\frac{1}{2}} - \beta^{n+3-\frac{1}{2}}}{\alpha-\beta} - \frac{1}{2}\right)\mathbf{h}\\
&= \left(\frac{\alpha^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{\beta^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{1}{2}\right) + \left(\frac{\alpha\cdot\alpha^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{\beta\cdot\beta^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{1}{2}\right)\mathbf{i}\\
&+ \left(\frac{\alpha^2\cdot\alpha^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{\beta^2\cdot\beta^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{1}{2}\right)\boldsymbol{\varepsilon} + \left(\frac{\alpha^3\cdot\alpha^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{\beta^3\cdot\beta^{n-\frac{1}{2}}}{\alpha-\beta} - \frac{1}{2}\right)\mathbf{h}\\
&= \frac{\alpha^{n-\frac{1}{2}}}{\alpha-\beta}\left(1+\alpha\mathbf{i}+\alpha^2\boldsymbol{\varepsilon}+\alpha^3\mathbf{h}\right) - \frac{\beta^{n-\frac{1}{2}}}{\alpha-\beta}\left(1+\beta\mathbf{i}+\beta^2\boldsymbol{\varepsilon}+\beta^3\mathbf{h}\right)\\
&- \frac{1}{2}\left(1+\mathbf{i}+\boldsymbol{\varepsilon}+\mathbf{h}\right),
\end{aligned}$$

which ends the proof. □

Using (9) and (5), one can easily prove the next theorem.

Theorem 4. *Let $n \geq 0$ be an integer. Then*

$$cH_n = \frac{1}{2}\alpha^{n-\frac{1}{2}}\hat{\alpha} + \frac{1}{2}\beta^{n-\frac{1}{2}}\hat{\beta}, \tag{12}$$

where α, β and $\hat{\alpha}$, $\hat{\beta}$ are given by (6) and (11), respectively.

Note that

$$\alpha + \beta = 6, \tag{13}$$

$$\alpha - \beta = 4\sqrt{2}, \tag{14}$$

$$\alpha\beta = 1. \tag{15}$$

By simple calculations we obtain

$$\begin{aligned}
\hat{\alpha}\hat{\beta} &= 1 - \alpha\beta + \alpha\beta(\alpha+\beta) + (\alpha\beta)^3 + (\alpha+\beta+\alpha\beta(\alpha+\beta)(\beta-\alpha))\mathbf{i}\\
&+ (\alpha^2+\beta^2+\alpha\beta(\alpha+\beta)(\beta-\alpha) + (\alpha\beta)^2(\alpha-\beta))\boldsymbol{\varepsilon}\\
&+ (\alpha^3+\beta^3+\alpha\beta(\alpha-\beta))\mathbf{h}.
\end{aligned}$$

By formulas (13)–(15) we get

$$\begin{aligned}
\alpha^2+\beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta = 34,\\
\alpha^3+\beta^3 &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = 198.
\end{aligned}$$

Thus

$$\hat{\alpha}\hat{\beta} = 7 + (6 - 24\sqrt{2})\mathbf{i} + (34 - 20\sqrt{2})\boldsymbol{\varepsilon} + (198 + 4\sqrt{2})\mathbf{h}. \tag{16}$$

In the same way we obtain

$$\hat{\beta}\hat{\alpha} = 7 + (6 + 24\sqrt{2})\mathbf{i} + (34 + 20\sqrt{2})\boldsymbol{\varepsilon} + (198 - 4\sqrt{2})\mathbf{h} \tag{17}$$

and

$$\begin{aligned} \hat{\mathbf{1}}\hat{\alpha} &= 117 + 82\sqrt{2} + (100 + 70\sqrt{2})\mathbf{i} + (32 + 22\sqrt{2})\boldsymbol{\varepsilon} + (86 + 60\sqrt{2})\mathbf{h}, \\ \hat{\alpha}\hat{\mathbf{1}} &= 117 + 82\sqrt{2} + (-92 - 66\sqrt{2})\mathbf{i} + (4 + 2\sqrt{2})\boldsymbol{\varepsilon} + (114 + 80\sqrt{2})\mathbf{h}, \\ \hat{\mathbf{1}}\hat{\beta} &= 117 - 82\sqrt{2} + (100 - 70\sqrt{2})\mathbf{i} + (32 - 22\sqrt{2})\boldsymbol{\varepsilon} + (86 - 60\sqrt{2})\mathbf{h}, \\ \hat{\beta}\hat{\mathbf{1}} &= 117 - 82\sqrt{2} + (-92 + 66\sqrt{2})\mathbf{i} + (4 - 2\sqrt{2})\boldsymbol{\varepsilon} + (114 - 80\sqrt{2})\mathbf{h}. \end{aligned} \tag{18}$$

Theorem 5 (General bilinear index-reduction formula for cobalancing hybrid numbers). *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then*

$$\begin{aligned} &bH_a \cdot bH_b - bH_c \cdot bH_d \\ &= \frac{\alpha^{c-\frac{1}{2}}\beta^{d-\frac{1}{2}} - \alpha^{a-\frac{1}{2}}\beta^{b-\frac{1}{2}}}{(\alpha-\beta)^2}\hat{\alpha}\hat{\beta} + \frac{\beta^{c-\frac{1}{2}}\alpha^{d-\frac{1}{2}} - \beta^{a-\frac{1}{2}}\alpha^{b-\frac{1}{2}}}{(\alpha-\beta)^2}\hat{\beta}\hat{\alpha} \\ &+ \frac{\alpha^{c-\frac{1}{2}} - \alpha^{a-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\alpha}\hat{\mathbf{1}} + \frac{\alpha^{d-\frac{1}{2}} - \alpha^{b-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\mathbf{1}}\hat{\alpha} + \frac{\beta^{a-\frac{1}{2}} - \beta^{c-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\beta}\hat{\mathbf{1}} + \frac{\beta^{b-\frac{1}{2}} - \beta^{d-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\mathbf{1}}\hat{\beta}, \end{aligned}$$

where α, β and $\hat{\alpha}\hat{\beta}$, $\hat{\beta}\hat{\alpha}$, $\hat{\mathbf{1}}\hat{\alpha}$, $\hat{\alpha}\hat{\mathbf{1}}$, $\hat{\mathbf{1}}\hat{\beta}$, $\hat{\beta}\hat{\mathbf{1}}$ are given by (6) *and* (16)–(18)*, respectively.*

Proof. By (10), we have

$$\begin{aligned} &bH_a \cdot bH_b - bH_c \cdot bH_d \\ &= \left(\frac{\alpha^{a-\frac{1}{2}}}{\alpha-\beta}\hat{\alpha} - \frac{\beta^{a-\frac{1}{2}}}{\alpha-\beta}\hat{\beta} - \frac{1}{2}\hat{\mathbf{1}}\right) \cdot \left(\frac{\alpha^{b-\frac{1}{2}}}{\alpha-\beta}\hat{\alpha} - \frac{\beta^{b-\frac{1}{2}}}{\alpha-\beta}\hat{\beta} - \frac{1}{2}\hat{\mathbf{1}}\right) \\ &- \left(\frac{\alpha^{c-\frac{1}{2}}}{\alpha-\beta}\hat{\alpha} - \frac{\beta^{c-\frac{1}{2}}}{\alpha-\beta}\hat{\beta} - \frac{1}{2}\hat{\mathbf{1}}\right) \cdot \left(\frac{\alpha^{d-\frac{1}{2}}}{\alpha-\beta}\hat{\alpha} - \frac{\beta^{d-\frac{1}{2}}}{\alpha-\beta}\hat{\beta} - \frac{1}{2}\hat{\mathbf{1}}\right) \\ &= -\frac{\alpha^{a-\frac{1}{2}}\beta^{b-\frac{1}{2}}}{(\alpha-\beta)^2}\hat{\alpha}\hat{\beta} - \frac{\alpha^{a-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\alpha}\hat{\mathbf{1}} - \frac{\beta^{a-\frac{1}{2}}\alpha^{b-\frac{1}{2}}}{(\alpha-\beta)^2}\hat{\beta}\hat{\alpha} + \frac{\beta^{a-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\beta}\hat{\mathbf{1}} \\ &- \frac{\alpha^{b-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\mathbf{1}}\hat{\alpha} + \frac{\beta^{b-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\mathbf{1}}\hat{\beta} + \frac{\alpha^{c-\frac{1}{2}}\beta^{d-\frac{1}{2}}}{(\alpha-\beta)^2}\hat{\alpha}\hat{\beta} + \frac{\alpha^{c-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\alpha}\hat{\mathbf{1}} \\ &+ \frac{\beta^{c-\frac{1}{2}}\alpha^{d-\frac{1}{2}}}{(\alpha-\beta)^2}\hat{\beta}\hat{\alpha} - \frac{\beta^{c-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\beta}\hat{\mathbf{1}} + \frac{\alpha^{d-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\mathbf{1}}\hat{\alpha} - \frac{\beta^{d-\frac{1}{2}}}{2(\alpha-\beta)}\hat{\mathbf{1}}\hat{\beta} \end{aligned}$$

and after calculations we get the result. □

Using (12), we can prove the next theorem.

Theorem 6 (General bilinear index-reduction formula for Lucas-cobalancing hybrid numbers). *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then*

$$\begin{aligned} &cH_a \cdot cH_b - cH_c \cdot cH_d \\ &= \left(\frac{1}{4}\alpha^{a-\frac{1}{2}}\beta^{b-\frac{1}{2}} - \frac{1}{4}\alpha^{c-\frac{1}{2}}\beta^{d-\frac{1}{2}}\right)\hat{\alpha}\hat{\beta} + \left(\frac{1}{4}\beta^{a-\frac{1}{2}}\alpha^{b-\frac{1}{2}} - \frac{1}{4}\beta^{c-\frac{1}{2}}\alpha^{d-\frac{1}{2}}\right)\hat{\beta}\hat{\alpha}, \end{aligned}$$

where α, β and $\hat{\alpha}\hat{\beta}$, $\hat{\beta}\hat{\alpha}$ are given by (6) *and* (16)–(17)*, respectively.*

For special values of a, b, c, d, by Theorems 5–6, we can obtain some identities for cobalancing and Lucas-cobalancing hybrid numbers:

- d'Ocagne type identity – for $a = n$, $b = m + 1$, $c = n + 1$, $d = m$,
- Vajda type identity – for $a = m + r$, $b = n - r$, $c = m$, $d = n$,
- first Halton type identity – for $a = m + r$, $b = n$, $c = r$, $d = m + n$,
- second Halton type identity – for $a = n + k$, $b = n - k$, $c = n + s$, $d = n - s$,
- Catalan type identity – for $a = n + r$, $b = n - r$, $c = d = n$,
- Cassini type identity – for $a = n + 1$, $b = n - 1$, $c = d = n$.

At the end, we give the generating functions for cobalancing and Lucas-cobalancing hybrid numbers.

Theorem 7. *The generating function for cobalancing hybrid number sequence $\{bH_n\}$ is*

$$g(t) = \frac{2\boldsymbol{\varepsilon} + 14\mathbf{h} + (2\mathbf{i} - 14\mathbf{h})\,t + (2 + 2\mathbf{h})\,t^2}{(1 - 6t + t^2)(1 - t)}.$$

Proof. Assume that the generating function of the cobalancing hybrid number sequence $\{bH_n\}$ has the form $g(t) = \sum_{n=0}^{\infty} bH_n t^n$. Then

$$g(t) = bH_0 + bH_1 t + bH_2 t^2 + \cdots .$$

Hence we get

$$\begin{aligned} -6t \cdot g(t) &= -6bH_0 t - 6bH_1 t^2 - 6bH_2 t^3 - \cdots \\ t^2 \cdot g(t) &= bH_0 t^2 + bH_1 t^3 + bH_2 t^4 + \cdots . \end{aligned}$$

By adding the above three equalities, we get

$$\begin{aligned} g(t)(1 - 6t + t^2) = bH_0 &+ (bH_1 - 6bH_0)\,t + (bH_2 - 6bH_1 + bH_0)\,t^2 \\ &+ (bH_3 - 6bH_2 + bH_1)\,t^3 + \cdots . \end{aligned}$$

As we know, $t^2+t^3+\cdots=\frac{t^2}{1-t}$, so adding this equality multiplied by $-2\cdot\hat{\mathbf{1}}$ to the above, we obtain

$$g(t)(1-6t+t^2)-\frac{2t^2\hat{\mathbf{1}}}{1-t}=bH_0+(bH_1-6bH_0)\,t$$
$$+\left(bH_2-6bH_1+bH_0-2\hat{\mathbf{1}}\right)t^2+\left(bH_3-6bH_2+bH_1-2\hat{\mathbf{1}}\right)t^3+\cdots$$

and we have

$$g(t)=\frac{\left[bH_0+(bH_1-6bH_0)\,t\right](1-t)+2t^2\hat{\mathbf{1}}}{(1-6t+t^2)(1-t)},$$

since $bH_n=6bH_{n-1}-bH_{n-2}+2\,(1+\mathbf{i}+\boldsymbol{\varepsilon}+\mathbf{h})=6bH_{n-1}-bH_{n-2}+2\hat{\mathbf{1}}$ (see Theorem 1) and the coefficients of t^n for $n\geq 2$ are equal to zero. Moreover, by simple calculations we have

$$g(t)=\frac{bH_0+(bH_1-7bH_0)\,t+\left(2\hat{\mathbf{1}}-bH_1+6bH_0\right)t^2}{(1-6t+t^2)(1-t)}$$

and

$$bH_1-7bH_0=2\mathbf{i}-14\mathbf{h},\quad 2\hat{\mathbf{1}}-bH_1+6bH_0=2+2\mathbf{h}.$$

□

In the same way we can prove the next result.

Theorem 8. *The generating function for Lucas-cobalancing hybrid number sequence* $\{cH_n\}$ *is*

$$G(t)=\frac{-1+\mathbf{i}+7\boldsymbol{\varepsilon}+41\mathbf{h}+(7+\mathbf{i}-\boldsymbol{\varepsilon}-7\mathbf{h})\,t}{1-6t+t^2}.$$

Concluding Remarks. In [8], the authors showed that cobalancing numbers are very closely related to the Pell sequence and the Lucas-cobalancing numbers – to the associated Pell sequence. The hybrid Pell numbers were introduced and studied in [10]. Using some relationships between cobalancing and Pell numbers, one can look for relationships between cobalancing and Pell hybrid numbers.

In [4], Özkoç generalized the cobalancing and Lucas-cobalancing numbers in the following way. Let b_n^k denote the nth k-cobalancing number and c_n^k denote the nth k-Lucas cobalancing number which are the numbers defined by

$$b_n^k=6kb_{n-1}^k-b_{n-2}^k+2 \text{ for } n\geq 2,\ \text{with } b_0^k=0, b_1^k=0,$$
$$c_n^k=6kc_{n-1}^k-c_{n-2}^k \text{ for } n\geq 2,\ \text{with } c_0^k=6k-7, c_1^k=1$$

for some integer $k\geq 1$. Similarly to the previous considerations, we have defined additionally $b_0^k=0$ and $c_0^k=6k-7$. For $k=1$ we obtain classical cobalancing numbers and Lucas-cobalancing numbers. Using these generalizations, we can define generalizations of cobalancing and Lucas-cobalancing hybrid numbers.

The results presented in this paper have the potential to motivate further researchers of the subject of generalizations of cobalancing hybrid numbers and links of cobalancing hybrid numbers with Pell hybrid numbers.

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