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HELLMANN POTENTIAL IN SPINLESS SALPETER EQUATION WITH POTENTIAL BARRIER WITHIN THE FRAMEWORK OF NIKIFOROV-UVAROV METHOD

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ABSTRACT

In this paper, we have solved the spinless Salpeter equation (SSE) with Hellmann potential under the framework of Nikiforov-Uvarov (NU) method. The energy eigenvalues and corresponding wave functions for this system express in terms of the Jacobi polynomial are also obtained. With the help of approximation scheme the potential barrier has been evaluated. The results obtained in this work would have many applications in nuclear physics, chemical physics, atomic and molecular physics, molecular chemistry and other related areas as the results under limiting cases could be used to study the binding energy and interaction of some diatomic molecules. As a guide to interested readers, we have provided numerical data which discuss the energy spectra for this system.

Keywords: Hellmann potential, spinless Salpeter equation, Nikiforov-Uvarov method, potential barrier.

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1. INTRODUCTION

There has been an increasing interest in finding the analytical solutions of wave equations in relativistic and non-relativistic quantum mechanics such as Schrödinger, Klein-Gordon, Dirac, Duffin Kemmer-Petian (DKF) and Spinless Bethe-Salpeter equation with different potential models [1-8]. The spinless Salpeter Equation (SSE) may be considered as a standard approximation to the Bethe-Salpeter formalism, neglecting the spin degree freedom [9]. The Bethe-Salpeter equation is the semi-relativistic equation that describes the bound states of a two body quantum field system in a relativistic covariant formalism [10]. The SSE is a generalization of Schrödinger equation in the quantum relativistic regime [11]. Consequently, many authors have resorted to approximation technique to deal with the problems arising from the SSE [12]. In recent times, many authors have investigated the SSE for various potential models, Hassanabadi et al., [13] studied the SSE with hyperbolic potential via the SUSYQM formalism. Zarrinkamar et al., [14] studied the two body Salpeter equation with exponential potential using SUSYQM method. Also, Ikot et al. [15] investigated the SSE with generalized Hulthen potential using the SUSYQM approach.

The superposition of the Coulomb plus Yukawa potential suggested by Hellmann is given by

$$V(r) = -\frac{V_0}{r} + \frac{V_1 e^{-\alpha r}}{r}, \tag{1}$$

where V_0 and V_1 are the strength of Coulomb and the Yukawa potentials respectively, and α is the screening parameter [14, 16]. The Hellmann potential has been used by many authors to represent the electron core [17, 18] or the electron ion [19, 20] interactions. Dutt et al., [21] investigated the bound state energies as well as wave functions of this potential, using the large-N expansion technique. Nesser and Abdelmonem, using the J-matrix approach, studied the trajectories of the poles of the S-matrix for Hellmann potential in the complex energy plane near the critical screening parameter [22].

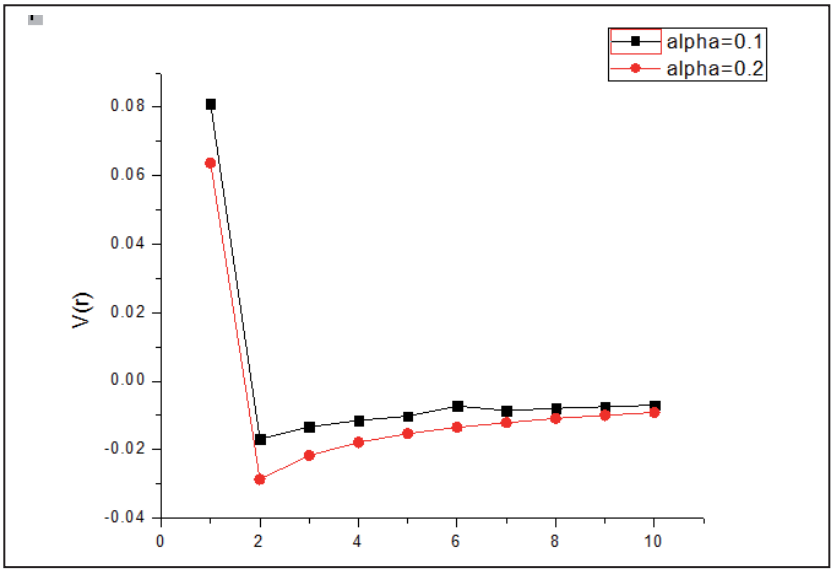

Figure 1: Behaviour of Hellmann potential $V(r)$ with position r

The behaviour of Hellmann potential with r for values of screening parameters – alpha $(\alpha) = 0.1$ and 0.2 with potential parameters $V_0 = 0.1$ and $V_1 = 0.2$ is presented in Fig. 1.

The aim of this work is to solve the SSE equation for the Hellmann potential and to calculate the energy eigenvalues and the corresponding wavefunctions which are expressed in terms of Jacobi polynomials for any arbitrary $l-$state using a suitable approximation scheme. The method employed is the Nikiforov and Uvarov method which is more suitable for obtaining exact or approximate solution of a second order differential equation through means of special orthogonal function [23].

2. THE NIKIFOROV-UVAROV (NU) METHOD

The NU method was presented by Nikiforov and Uvarov [24] and has been employed to solve second order differential equations such as Schrödinger wave equation (SWE), Klein-Gordon equation (KGE), Dirac equation (DE) etc.

The SSE can be solved by transforming it into a hypergeometric type equation using the transformation $s = s(x)$ and then solved systematically. Its resulting equation is expressed as:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of at most second degree and $\tilde{\tau}(s)$ is a polynomial of at most first degree and $\psi(s)$ is a wavefunction of the hypergeometric type equation. The parametric generalization of the NU method is given by the generalized hypergeometric type equation [23].

$$\psi''(s)+\frac{(c_1-c_2 s)}{s(1-c_3 s)}\psi'(s)+\frac{1}{s^2(1-c_3 s)^2}\left[-\xi_1 s^2+\xi_2 s-\xi_3\right]\psi(s)=0 \tag{3}$$

Eq. (3) is solved by comparing it with Eq. (2) and the following polynomials are obtained

$$\tilde{\tau}(s)=(c_1-c_2 s),\ \sigma(s)=s(1-c_3 s),\ \tilde{\sigma}(s)=-\xi_1 s^2+\xi_2 s-\xi_3 \tag{4}$$

According to NU method, the energy eigenvalue equation and eigenfunction respectively satisfy the following sets of equation,

$$c_2 n-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0 \tag{5}$$

$$\psi(s)=N_{nl}s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{\left(c_{10}-1,\ \frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s), \tag{6}$$

where

$$\begin{aligned}
&c_4=\frac{1}{2}(1-c_1);\ c_5=\frac{1}{2}(c_2-2c_3);\ c_6=c_5^2+\xi_1\\
&c_7=2c_4c_5-\xi_2;\ c_8=c_4^2+\xi_3;\ c_9=c_3c_7+c_3^2c_8+c_6\\
&c_{10}=c_1+2c_4+2\sqrt{c_8};\ c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)\\
&c_{12}=c_4+\sqrt{c_8};\ c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)
\end{aligned} \tag{7}$$

3. TWO-BODY SPINLESS SALPETER EQUATION

The SSE for two-body particles interacting in a spherically symmetric potential in the centre of mass system appears as [9, 10, 15]

$$\left[\sum_{i=1,2}\left(\sqrt{-\nabla^2+m_i^2}-m_i\right)+V(r)-E_{nl}\right]\chi(r)=0, \tag{8}$$

where

$$\chi(r) = R_{nl}(r)\, Y_{lm}(\theta, \phi) \tag{9}$$

For heavy interacting particles and using suitable transformation (see Refs [10,11]), one can recast SSE of Eq. (8) as

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{l(l+1)}{2\mu r^2} + W_{nl}(r) - \frac{W_{nl}^2(r)}{2\tilde{m}} \right] \psi_{nl}(r) = 0\,, \tag{10}$$

where

$$\left.\begin{aligned} W_{nl}(r) &= V(r) - E_{nl} \quad ; \Im = \mu\left(\frac{m_1 m_2}{m_1 m_2 - 3\mu^2}\right)^{\frac{1}{3}} \\ \mu &= \frac{m_1 m_2}{m_1 + m_2} \quad ; \tilde{m} = \frac{\Im^3}{\mu^2} = \frac{m_1 m_2 \mu}{m_1 m_2 - 3\mu^2} \end{aligned}\right\} \tag{11}$$

We consider the Hellmann potential defined as in Eq. (1) and substituting it into Eq. (10) and using the transformation $s = e^{-\alpha r}$ yields

$$\alpha^2\left[s^2\frac{d^2\psi_{nl}}{ds^2} + s\frac{d\psi_{nl}}{ds} \right] + \frac{2\mu}{\hbar^2 r^2}\left\{ \begin{aligned} &\frac{V_1^2 s^2}{2\tilde{m}} + \left[V_1 r + \frac{2V_0 V_1}{\tilde{m}} + \frac{V_1 r E_{nl}}{2\tilde{m}} \right] s \\ &+ \left[V_0 r + r^2 E_{nl} + \frac{(V_0 + rE_{nl})^2}{2\tilde{m}} - \frac{l(l+1)}{2\mu} \right] \end{aligned} \right\} \tag{12}$$

Since the SSE with Hellmann potential has no exact solution we use an approximation for the centrifugal term as [25]

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\left(1 - e^{-\alpha r}\right)^2} \tag{13}$$

and hence Eq. (12) becomes

$$\frac{d^2\psi_{nl}}{ds^2} + \frac{1-s}{s(1-s)}\frac{d\psi_{nl}}{ds} + \frac{1}{s^2(1-s)^2}\left[-As^2 + Bs - C \right]\psi_{nl} = 0\,, \tag{14}$$

where

$$\left.\begin{aligned}
\xi_1 = A &= \left[\frac{V_1}{\alpha}\left(1+\frac{E_{nl}}{2\widetilde{m}}\right)-\frac{E_{nl}}{\alpha^2}\left(1+\frac{E_{nl}}{2\widetilde{m}}\right)-\frac{V_1^2}{2\widetilde{m}}\right]\frac{2\mu}{\hbar^2} \\
\xi_2 = B &= \left[\frac{V_1}{\alpha}\left(1+\frac{E_{nl}}{2\widetilde{m}}\right)-\frac{V_0}{\alpha}\left(1+\frac{E_{nl}}{\widetilde{m}}\right)-\frac{E_{nl}}{\alpha^2}\left(2+\frac{E_{nl}}{\widetilde{m}}\right)+\frac{V_0V_1}{\widetilde{m}}\right]\frac{2\mu}{\hbar^2} \\
\xi_3 = C &= \frac{2\mu}{\hbar^2}\left[\frac{l(l+1)}{2\mu}-\frac{(V_0+E_{nl})^2}{2\widetilde{m}}-\frac{1}{\alpha^2}(V_0\alpha+E_{nl})\right]
\end{aligned}\right\} \quad (15)$$

Comparing Eq. (4) with Eq. (3), we obtain the following parameters

$$\begin{aligned}
&c_1 = c_2 = c_3 = 1; c_4 = 0;\ \ c_5 = -\frac{1}{2} \\
&c_6 = \frac{2\mu}{\hbar^2}\left[\frac{V_1}{\alpha}\left(1+\frac{E_{nl}}{\widetilde{m}}\right)-\frac{E_{nl}}{\alpha^2}\left(1+\frac{E_{nl}}{\widetilde{m}}\right)-\frac{V_1^2}{2m}\right]+\frac{1}{4} \\
&c_7 = \frac{2\mu}{\hbar^2}\left[\frac{V_0}{\alpha}\left(1+\frac{E_{nl}}{\widetilde{m}}\right)+\frac{E_{nl}}{\alpha^2}\left(2+\frac{E_{nl}}{\widetilde{m}}\right)-\frac{V_1}{\alpha}\left(1+\frac{E_{nl}}{2\widetilde{m}}\right)-\frac{V_0V_1}{\widetilde{m}}\right] \\
&c_8 = \beta = \frac{2\mu}{\hbar^2}\left[\frac{(V_0+E_{nl})^2}{2\widetilde{m}}+\frac{1}{\alpha^2}(V_0\alpha+E_{nl})-\frac{2(l+1)}{2\mu}\right] \\
&c_9 = \gamma = \frac{2\mu}{\hbar^2}\left[\frac{(V_0+E_{nl})^2}{2\widetilde{m}}+\frac{2}{\alpha^2}(V_0\alpha+E_{nl})-\frac{V_1}{2\widetilde{m}}(2V_0+V_1)+\frac{V_0E_{nl}}{\alpha\widetilde{m}}-\frac{l(l+1)}{2\mu}\right]+\frac{1}{4} \\
&c_{10} = 2\sqrt{\beta}; c_{11} = 2\left(1+\sqrt{\gamma}+\sqrt{\beta}\right),\ c_{12} = \sqrt{\beta}; c_{13} = -\frac{1}{2}-\left(\sqrt{\gamma}+\sqrt{\beta}\right).
\end{aligned} \quad (16)$$

Substituting Eq. (16) into the energy eigenvalue equation, we obtain the energy spectrum for the system as

$$\begin{aligned}
&\left(\frac{1}{\alpha^2}+\frac{1}{2}\right)E_{nl}^2-\left(\frac{V_0}{\alpha}+\frac{2\widetilde{m}}{\alpha^2}+\frac{1}{2}+\frac{1}{\alpha^2}\right)E_{nl} \\
&\qquad +(2n+1)\left(\sqrt{\gamma}+\sqrt{\beta}\right)+2\sqrt{\gamma\beta}+(n+l+1)^2+\frac{(V_0+V_1)}{\alpha}=0
\end{aligned} \quad (17)$$

Equation (17) can be solved explicitly for the energy eigenvalue as:

$$E_{nl}^{\pm}=\frac{1}{(\alpha^2+2)}\left(\alpha V_0+2\tilde{m}+\frac{\alpha^2}{2}+1\right)\pm\frac{\alpha^2}{\alpha^2+2}\sqrt{\left(\frac{V_0}{\alpha}+\frac{2\tilde{m}}{\alpha^2}+\frac{1}{\alpha^2}+\frac{1}{2}\right)^2-4\left(\frac{1}{\alpha^2}+\frac{1}{2}\right)\times\left[2n+1\left(\sqrt{\gamma}+\sqrt{\beta}\right)+2\sqrt{\gamma\beta}+(n+l+1)^2+\frac{V_0+V_1}{\alpha}\right]} \tag{18}$$

Using Eq. (16) and Eq. (6), the wavefunction for the system is obtained as

$$\psi_{nl}(s)=N_{nl}S^{\sqrt{\beta}}(1-s)^{\sqrt{\gamma}+\frac{1}{2}}P_n^{\left(2\sqrt{\beta}-1,2\sqrt{\gamma}+1\right)}(1-2s) \tag{19}$$

or equivalently, using $s=e^{-\alpha r}$

$$\psi_{nl}=N_{nl}e^{-\alpha\sqrt{\beta}\gamma}\left(1-e^{-\alpha r}\right)^{\sqrt{\gamma}+\frac{1}{2}}P_n^{\left(2\sqrt{\beta}-1,\,2\sqrt{\gamma}+1\right)}\left(1-2e^{-\alpha r}\right) \tag{20}$$

where N_{nl} is a normalization constant.

Setting $V_1=0$, the potential in Eq. (1) reduces to Coulomb potential and the corresponding energy and wavefunction are obtained by substituting this parameter into Eqs (18) and (22) respectively. Interestingly, if V_0 is set to zero $(V_0=0)$, the potential in Eq. (1) becomes Yukawa potential [26] which has many applications in physics and in chemistry.

The real numerical energy eigenvalues with $m=\hbar=1$, $V_0=0.1, V_2=0.2$ and $\mu=0.2$ for different values of $\alpha=1.0, 2.0, 3.0, 4.0\ and\ 5.0$ for this system are computed and presented in Tables 1–5 respectively for different quantum states of n and l. As expected, from the computed result, it can be seen that the energy does not degenerate. That is, there is no degeneracy.

Table 1: Energy eigenvalues $\alpha=1.0$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	3.83520122448	**0**	**1**	3.25263530927	**0**	**2**	1.29131220154	**0**	**3**	3.07205795833
1	**0**	1.83106642467	**1**	**1**	1.33365915141	**1**	**2**	2.68766788511	**1**	**3**	1.42957983415
2	**0**	2.0370629491	**2**	**1**	2.41383442312	**2**	**2**	2.86998068888-	**2**	**3**	2.79875592271
3	**0**	2.27836200590	**3**	**1**	2.55180422130	**3**	**2**	3.35184255355	**3**	**3**	3.09588134516
4	**0**	2.68186771011	**4**	**1**	3.00713612955	**4**	**2**	3.73963758052	**4**	**3**	3.61399171896
5	**0**	3.12022213532	**5**	**1**	3.39907223898	**5**	**2**	4.16137462817	**5**	**3**	4.03379417671
6	**0**	3.57905302049	**6**	**1**	3.83486836528	**6**	**2**	4.58013837976	**6**	**3**	4.46635207018
7	**0**	4.04269130263	**7**	**1**	4.27020920787	**7**	**2**	5.0077319940	**7**	**3**	4.88910471213
8	**0**	4.50957487726	**8**	**1**	4.71551921270	**8**	**2**	3.07205795833	**8**	**3**	5.31406934416

Table 2: Energy eigenvalues $\alpha = 2.0$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	6.08257238087	**0**	**1**	6.48113874570	**0**	**2**	6.38291443217	**0**	**3**	6.29773944695
1	**0**	4.74676725473	**1**	**1**	4.97491530907	**1**	**2**	4.67128431372	**1**	**3**	4.39669525191
2	**0**	2.02776579826	**2**	**1**	2.16157489144	**2**	**2**	1.80020066611	**2**	**3**	1.48900029723
3	**0**	2.29843058364	**3**	**1**	2.47208369082	**3**	**2**	2.86598475525	**3**	**3**	3.20071570551
4	**0**	3.10954479140	**4**	**1**	3.31929259014	**4**	**2**	3.38908305603	**4**	**3**	3.45355195380
5	**0**	3.71648006970	**5**	**1**	3.89037780843	**5**	**2**	4.15898882694	**5**	**3**	4.41090061829
6	**0**	4.48229883019	**6**	**1**	4.64314077843	**6**	**2**	4.81321144773	**6**	**3**	4.99046270993
7	**0**	5.20302830597	**7**	**1**	5.34027191417	**7**	**2**	5.54839101123	**7**	**3**	5.76836081323
8	**0**	5.96795908257	**8**	**1**	6.09194852307	**8**	**2**	6.26213424096	**8**	**3**	6.45408609213

Table 3: Energy eigenvalues $\alpha = 3.0$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	10.2350299832	**0**	**1**	9.35180924373	**0**	**2**	9.19460253990	**0**	**3**	9.05268948158
1	**0**	11.2220909800	**1**	**1**	10.3651118631	**1**	**2**	10.0257628074	**1**	**3**	9.68673106388
2	**0**	9.81194594005	**2**	**1**	8.96387393108	**2**	**2**	8.48598942118	**2**	**3**	8.02208536232
3	**0**	6.48503078461	**3**	**1**	5.75182961724	**3**	**2**	5.27646404919	**3**	**3**	4.82660065724
4	**0**	2.33867244344	**4**	**1**	1.81887622436	**4**	**2**	1.42039538028	**4**	**3**	1.02908936200
5	**0**	2.99812434008	**5**	**1**	3.39679669331	**5**	**2**	3.78655172573	**5**	**3**	4.187004046026
6	**0**	3.85530200004	**6**	**1**	3.71053799913	**6**	**2**	3.63252578142	**6**	**3**	3.56077737888
7	**0**	4.61394448601	**7**	**1**	4.80455275085	**7**	**2**	5.02504554292	**7**	**3**	5.26796816372
8	**0**	5.45934768824	**8**	**1**	5.44662939502	**8**	**2**	5.48375214126	**8**	**3**	5.53600395519

Table 4: Energy eigenvalues $\alpha = 4.0$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	13.7006867392	**0**	**1**	11.8430575071	**0**	**2**	20.9427784432	**0**	**3**	14.2899710849
1	**0**	18.3371888966	**1**	**1**	16.6704924217-	**1**	**2**	24.2985909568-	**1**	**3**	18.4459914909
2	**0**	24.6488977652	**2**	**1**	18.4114357902	**2**	**2**	24.8097354084	**2**	**3**	19.5609820183

3	**0**	26.1086291751	**3**	**1**	23.4152491384	**3**	**2**	27.3029083425	**3**	**3**	18.0976167937
4	**0**	22.5806208677	**4**	**1**	27.7562879739	**4**	**2**	32.3975714404	**4**	**3**	14.5533141388
5	**0**	17.2146922061	**5**	**1**	32.9888932858	**5**	**2**	35.8129641188	**5**	**3**	9.69369691662
6	**0**	10.6573138361	**6**	**1**	37.7155209194	**6**	**2**	26.2562365864	**6**	**3**	4.47905821332
7	**0**	3.99270445261	**7**	**1**	25.9993801789	**7**	**2**	16.1323997767	**7**	**3**	1.79295968361
8	**0**	3.34818734801	**8**	**1**	14.2788506023	**8**	**2**	6.49461746049	**8**	**3**	5.00562587332

Table 5: Energy eigenvalues for $\alpha = 5.0$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	18.2587083845	**0**	**1**	74.6286992089	**0**	**2**	80.0385081664	**0**	**3**	68.7732793484
1	**0**	27.2997816966	**1**	**1**	66.0689022770	**1**	**2**	70.7768372287	**1**	**3**	64.2334013039
2	**0**	32.0492522899	**2**	**1**	72.3355675115	**2**	**2**	76.3473333496	**2**	**3**	67.9592868635
3	**0**	38.5074264307	**3**	**1**	67.6358625403	**3**	**2**	71.1302064479	**3**	**3**	66.2550370077
4	**0**	43.8191331531	**4**	**1**	73.9650827497	**4**	**2**	67.7356120163	**4**	**3**	70.6998062586
5	**0**	50.3628866270	**5**	**1**	72.1369190840	**5**	**2**	67.9121807027	**5**	**3**	71.1056191688
6	**0**	56.0201413341	**6**	**1**	64.5193497779	**6**	**2**	73.8717299348	**6**	**3**	76.0773618533
7	**0**	62.5745921512	**7**	**1**	69.6658426891	**7**	**2**	75.6693106173	**7**	**3**	78.0471624055
8	**0**	68.4555728638	**8**	**1**	74.7978095668	**8**	**2**	63.0939017308	**8**	**3**	64.9667457199

4. CONCLUSION

In this research paper, we have solved the two-body Spinless Salpeter equation with Hellmann potential using a powerful approach; the Nikiforov-Uvarov method. With the help of a good approximation scheme, the centrifugal term (potential barrier) encountered in the problem has been evaluated. Explicitly, we have obtained the bound state energy eigenvalues and the corresponding unnormalized eigenfunction (wave function) expressed in terms of Jacobi polynomial. Numerical data for the energy spectrum for this system are discussed, indicating usefulness for other physical systems.

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