Two hierarchies of \( R \)-recursive functions

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**Abstract**

In the paper some aspects of complexity of \( R \)-recursive functions are considered. The limit hierarchy of \( R \)-recursive functions is introduced by the analogy to the \( \mu \)-hierarchy. Then its properties and relations to the \( \mu \)-hierarchy are analysed.

1. **Introduction**

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals \( R \) (called \( R \)-recursive functions) in the analogous way to the classical recursive functions on the natural numbers \( \mathbb{N} \). His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore’s model has the zero-finding operation \( \mu \), which is used to construct \( \mu \)-hierarchy of \( R \)-recursive functions.

It was shown [5] that the zero-finding operator \( \mu \) can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to \( \mu \)-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called $R$-recursive functions [4].

**Definition 2.1** The set of $R$-recursive functions is generated from the constants $0,1$ by the operations:

1) composition: $h(\bar{x}) = f\left(g\left(\bar{x}\right)\right)$;

2) differential recursion: $h(\bar{x},0) = f\left(\bar{x}\right), \partial_y h(\bar{x},y) = g\left(\bar{x},y,h(\bar{x},y)\right)$ (the equivalent formulation can be given by integrals: $h(\bar{x},y) = f\left(\bar{x}\right) + \int_0^y g\left(\bar{x},y',h(\bar{x},y')\right)dy'$);

3) $\mu$-recursion $h(\bar{x}) = \mu_y f\left(\bar{x},y\right) = \inf\left\{y : f\left(\bar{x},y\right) = 0\right\}$, where infimum chooses the number $y$ with the smallest absolute value and for two $y$ with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if $h$ is defined by a differential recursion then $h$ is defined only where a finite and unique solution exists. This is why the set of $R$-recursive functions includes also partial functions. We use (after [4]) the name of $R$-recursive functions in the article, however we should remember that in reality we have partiality here (partial $R$-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive $y$ or just below some negative $y$ then the infimum operation returns that $y$ even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore’s definition [4] $\mu$-operation is replaced by infinite limits: $h(\bar{x}) = \liminf_{y\to\infty} g\left(\bar{x},y\right)$, $h(\bar{x}) = \limsup_{y\to\infty} g\left(\bar{x},y\right)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(\bar{x}) = \lim_{y\to\infty} g\left(\bar{x},y\right)$, which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of R-recursive functions is closed under the operations of infinite limits: \( h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x}, y) \), \( h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x}, y) \), \( h(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y) \).

3. Hierarchies

The operator \( \mu \) is a key operator in generating the R-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of \( \mu \) in the definition of a given \( f \).

Definition 3.1 ([4]) For a given R-recursive expression \( s(\bar{x}) \), let \( M_x(s) \) (the \( \mu \)-number with respect to \( x_i \)) be defined as follows:

\[
M_x(0) = M_x(1) = M_x(-1) = 0, \quad (1)
\]

\[
M_x(f(g_1, g_2, \ldots)) = \max \left( M_{x_j}(f) + M_x(g_j) \right), \quad (2)
\]

\[
M_x(h = f + \int_0^y g(\bar{x}, y', h) \, dy') = \max \left( M_x(f), M_x(g), M_h(g) \right), \quad (3)
\]

\[
M_y(h = f + \int_0^y g(\bar{x}, y', h) \, dy') = \max \left( M_y(g), M_h(g) \right), \quad (4)
\]

\[
M_x(\mu, f(\bar{x}, y)) = \max \left( M_x(f), M_y(f) \right) + 1, \quad (5)
\]

where \( x \) can be any \( x_1, \ldots, x_n \) for \( \bar{x} = (x_1, \ldots, x_n) \).

For an R-recursive function \( f \), let \( M(f) = \max_x M_x(s) \) minimized over all expressions \( s \) that define \( f \). Now we are ready to define M-hierarchy (\( \mu \)-hierarchy) as a family of \( M_j = \{ f : M'(f) \leq j \} \).

Let us construct the analogous definition of L-hierarchy by replacing in the above definition \( M_x \) by \( L_x \) and changing line (5) to the following form (5‘):

\[
L_x\left( \liminf_{y \to \infty} g(\bar{x}, y) \right) = L_x\left( \limsup_{y \to \infty} g(\bar{x}, y) \right) =
\]

\[
= L_x\left( \lim g(\bar{x}, y) \right) = \max \left( L_x(f), L_y(f) \right) + 1.
\]

For an R-recursive function \( f \), let \( L(f) = \max_x L_x(s) \) minimized over all expressions \( S \) that define \( f \) without using the \( \mu \)-operation.
**Definition 3.2** The $L$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

**Lemma 3.3** The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $R$-recursive function. After Moore [4] we can conclude that such functions as: $-x, x + y, xy, x/y, e^x, \ln x, y^x, \sin x, \cos x$ are primitive $R$-recursive.

We can give a few results on some levels of the limit hierarchy.

**Lemma 3.4.** The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

**Proof.** It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases}$ we obtain $\operatorname{sgn}(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2 \arctan 1}$ and $|x| = \operatorname{sgn}(x)x$.

We should be careful with definitions of functions by cases:

**Lemma 3.5** For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ M & \text{if } f(\bar{x}) \geq k - 1 \end{cases}$ and $g_i \in L_i$ for all $1 \leq i \leq k$,

If $f \in L_m$ the function $h$ belongs to $L_{\max(n_1, n_2, \ldots, n_m + 1)}$. 
Proof. Let us see that \( eq(x, y) = \delta(x - y) \in L_1 \) and 
\[
geq(x, y) = \frac{\left(\text{sgn}(x - y) + eq(x, y)\right)}{2} + \frac{1}{2} \in L_1.
\]
Then of course 
\[
h(\bar{x}) = \sum_{i=1}^{k} g_i(\bar{x})eq(f(\bar{x}), i-1) + g_k(\bar{x})ge(f(\bar{x}), k-1) \square
\]
Of course this result can be easily extended to other forms of definitions by cases.

**Lemma 3.6** The function \( \Theta(x) \) (equal to 1 if \( x \geq 0 \), otherwise 0), maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n+1] \) and \( p(x) = 0 \) for \( x \in [2n+1, 2n+2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

**Proof.** We give the proper definitions (from [6]) for these functions. Let 
\[
\Theta(x) = \delta(x - |x|),
\]
\[
\max(x, y) = x\delta(x - y) + (1 - \delta(x - y))\left[x\Theta(x - y) + y\Theta(y - x)\right],
\]
\[
s(x) = \Theta(\sin(\pi x)).
\]
The function \( p(x) \) can be given as 
\[
s(x)\left(1 - \delta\left(\sin\left(\frac{x-1}{2}\right)\pi\right)\right), \text{ so } p \in L_2.
\]
The floor function we can define by the auxiliary function \( w(0) = 0, \partial_x w(x) = 2\Theta(-\sin(2\pi x)) \) as
\[
\lfloor x \rfloor = \begin{cases} 
2w(x/2) & \text{if } p(x) = 1, \\
2w((x-1)/2) & \text{if } p(x) = 0.
\end{cases}
\]
From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \) \( \square \)

Let us recall that if \( f : R^n \to R \) is an \( R \)-recursive function then the function \( f_{\text{iter}}(i, \bar{x}) \) is \( R \)-recursive, too.

**Lemma 3.7** Let \( f : R^n \to R \) belongs to the class \( L_i \), then we have \( f_{\text{iter}} : R^{n+1} \to R \) is in \( L_{\max(2, j)} \).

**Proof.** The definitions, which were given by Moore [3] \( f_{\text{iter}}(i, \bar{x}) = h(2i) \), where 
\[
h(0) = g(0) = \bar{x},
\]
\[ \partial_t g(t) = \left[ f(h(t)) - h(t) \right] s(t), \]
\[ \partial_t h(t) = \frac{g(t) - h(t)}{r(t)} (1 - s(t)), \]

with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0 \), \( \partial_t r(t) = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( R^l \)-recursive functions \( \gamma_2 : R^2 \to R \), \( \gamma'_2, \gamma''_2 : R \to R \) such that \( (\forall x,y \in R) \gamma'_2 (\gamma_2(x,y)) = x \), \( (\forall x,y \in R) \gamma''_2 (\gamma_2(x,y)) = y \), have the following properties: \( \gamma_2, \gamma'_2 \) are in \( L_{10} \), \( \gamma''_2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2, \Gamma'_2, \Gamma''_2 \), which are coding and decoding functions in the interval \( (0,1): \Gamma_2(x,y) = c(x)+c(y)/10 \), where
\[ c(x) = \lim_{i \to \infty} z(a(i,x))/10^i + b(i,x)/10^i, \]
and later \( z(i) = \lim_{i \to \infty} z_{iter}^*(i,x), \)
\[ z_{iter}^*(i,a_1...a_n,a_{n+1}...) = a_1...a_n0...a_{n+1}0.a_{n+2}... \]
\[ a(i,0.a_1a_2...a_i...) = 0.a_1...a_i \]
\[ b(i,0.a_1a_2...a_i...) = 0.0.a_1...a_i \]

\[ (z'(x) = \begin{cases} 100(x) + 10(x-[x]), & \text{if } [x] \neq x, \\ x, & \text{if } [x] = x \end{cases} \]

\( z' \) belongs to \( L_4 \), hence \( \Gamma_2(x,y) \in L_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma'_2(x) \) is in \( L_{10} \), but \( \Gamma''_2(x) = \Gamma'_2(10-[10x]) \) so \( \Gamma''_2 \in L_{14} \).

The functions \( \Gamma_2, \Gamma'_2, \Gamma''_2 \) can be extended to all reals by one-to-one \( f : (0,1) \to R \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \to R \) and \( \gamma'_n : R \to R \) for \( i = 1,...,n \) such that
\[ (\forall i)(\forall x_1,...,x_n \in R) \gamma'_n(\gamma_n(x_1,...,x_n)) = x_i \]
in the same class: \( \gamma_n, \gamma'_n \in L_{10} \) and \( (\forall i > 1) \gamma'_n \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant \( p \in N \) that for the function

\[
\prod_{z=0}^{y} f(\bar{x}, z) = \begin{cases} 
\prod_{y=0}^{0} f(x_0,0) f(x_1,0) \ldots f(x_{y-1},0), & \text{if } y \geq 1, \\
1, & \text{if } 0 \leq y < 1, \\
0, & \text{if } y < 0,
\end{cases}
\]

if the function \( f \) is in the class \( L_m \) then \( \prod_{z=0}^{y} f(\bar{x}, z) \) is in the class \( L_{m+p} \) (\( p \) is independent of \( m \)).

Proof. By the definitions

\[
t(w) = \gamma_{n+2} \left( \gamma_{n+2}^{1,1}(w), \gamma_{n+2}^{1,2}(w) + 1, f(\gamma_{n+2}^{1,1}(w), \gamma_{n+2}^{1,2}(w)) \gamma_{n+2}^{1,2}(w) \right)
\]

and

\[
S(\bar{x}, z) = t_{(z, f(s(\bar{x}, 0)) \ldots )} = t_{\text{iter}(\left[ z \right], \gamma_{n+2}^{1,2}(\bar{x}, 0, 1))}
\]

we get the property

\[
\prod_{y=0}^{z} f(\bar{x}, y) = \gamma_{n+2}^{1,2}(S(\bar{x}, z)).
\]

From the definition of the limit hierarchy we get \( \prod_{y=0}^{z} f(\bar{x}, y) \in L_{m+38} \).

In the rest of the paper we will use the constant \( p \) as the number of limits used in the recursive definition of the product \( \prod_{y=0}^{z} f(\bar{x}, y) \) instead of the value 38.

The above constructions are tedious and can be improved with a better approximation of \( p \).

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between \( L \)-hierarchy and \( M \)-hierarchy.

Theorem 4.1 Let \( f : R^n \to R \) be an \( R \)-recursive function. Then if \( f \in L_i \) then \( f \in M_{10i} \).

Proof. We use a simple induction here. The case \( i = 0 \) is given in Lemma 3.3. Now let us suppose that the thesis is true for \( i = n \). Let \( f \in L_{n+1} \) be defined as \( f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y) \) for \( g \in L_n \). Then we can recall Theorem 4.2 from [6] which gives us the following result: to define \( f \) from \( g \) it is necessary to use at
most 10 μ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \).

Similar inferences hold for \( \lim \inf \), \( \lim \sup \).

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(x,y): R^{n+1} \to R \) is in the class \( L_n \) then the function \( g: R^n \to R \), \( g(x) = \mu_y f(x,y) \) is in the class \( L_{n+3p+9} \) is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function \( g(x,y) = \mu_y f(x,y) \) for \( f(x,y): R^{n+1} \to R \) (\( f \) - \( R \)-recursive) replacing the \( \mu \)-operator by limit operation. First we introduced the function

\[
Z^f(\bar{x}, z) = \begin{cases} 
\inf_y \{ f : K^f(\bar{x}, y) = 0 \}, & \text{if } z = 0 \text{ and } \exists y K^f(\bar{x}, y) = 0, \\
\text{undefined} & \text{if } z = 0 \text{ and } \forall y K^f(\bar{x}, y) \neq 0, \\
1 & \text{if } z \neq 0,
\end{cases}
\]

given in the following way:

\[
Z^f(\bar{x}, z) = \begin{cases} 
\text{undefined} & \text{if } (z = 0) \land (S^f(\bar{x}) < 1/12), \\
\sqrt{S^f(\bar{x}) - 1/12} & \text{if } (z = 0) \land (S^f(\bar{x}) \geq 1/12) \land f(\bar{x}, \sqrt{S^f(\bar{x}) - 1/12}) = 0, \\
-\sqrt{S^f(\bar{x}) - 1/12}, & \text{if } (z = 0) \land (S^f(\bar{x}) \geq 1/12) \land f(\bar{x}, -\sqrt{S^f(\bar{x}) - 1/12}) = 0, \\
1, & \text{if } z \neq 0,
\end{cases}
\]

where \( S^f(\bar{x}) = \lim_{t \to \infty} S^f_i(\bar{x}, t) + \lim_{t \to \infty} S^f_i(\bar{x}, t) \). Both functions \( S^f_1, S^f_2 \) are defined by an integration

\[
S^f_i(\bar{x}, t) = \int y^i \left( 1 - h^f(\bar{x}, (-1)^{i+1} y - 1/2, (-1)^{i+1} y + 1/2) \right) dy, \quad i = 1, 2
\]

from \( h^f(\bar{x}, a, b) = \liminf_{y \to \infty} \prod_{w=0}^{x+1} K^f(\bar{x}, a + w \frac{b-a}{z}) \) where \( K^f \) is the characteristic function of \( f \).

Hence we can conclude that if \( K^f \) is in the \( L_s \) then \( Z^f \) is in the class \( L_{s+p+3} \).

Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5].
\[ K^f_\mu (y) = 1 - \lim_{a \to -\infty} \lim_{b \to +\infty} \lim_{z \to +\infty} G^f_\mu (\bar{x}, z, a, b, y), \]

where \( G^f_\mu (\bar{x}, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{[z]}\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a, b]\) and value 0 otherwise. Precisely for \(y\) from \([a, a + \frac{b-a}{2^{[z]}}]\)

\[ G^f_\mu (\bar{x}, z, a, b, y) = \begin{cases} 1, & \text{if } h^f (\bar{x}, a, a + \frac{b-a}{2^{[z]}}) = 0, \\ 0, & \text{otherwise} \end{cases} \]

for \(y \in \left(a + \frac{(k-1)(b-a)}{2^{[z]}}, a + \frac{k(b-a)}{2^{[z]}}\right)\) (where \(k = 2, 3, \ldots, 2^n\)) we have:

\[ G^f_\mu (\bar{x}, z, a, b, y) = \begin{cases} 1, & \text{if } \prod_{i=1}^{k-1} h^f (\bar{x}, a + \frac{(i-1)(b-a)}{2^{[z]}}, a + \frac{i(b-a)}{2^{[z]}}) \neq 0, \\ 0, & \text{otherwise} \end{cases} \]

and for \(Y \notin [A, B]\) the function \(g^f_\mu\) is equal to 2.

The definition of \(G_f\) is given by the cases with respect to the value of the expression given by \(\prod h^f\), since for \(f \in L_m\), the function \(h_f \in L_{m+p+2}\) and \(G^f \in L_{m+2+p+3}\). Then we have \(K^f_\mu \in L_{m+2+p+6}\). Now we must use the function \(K^f_\mu\) in the same way as \(K^f\) which gives us \(Z_f\) in the class \(L_{m+3+p+9}\). The final definition of \(g(\bar{x}) = \mu_f(\bar{x}, y)\) ([5] Theorem 4.3) given below
\[ g(x) = \begin{cases} 
Z^{f'}(x,0) - Z^{f'}(x,0), & \text{if } S^{f'}(x) < \frac{1}{12} \land S^{f'}(x) < \frac{1}{12}, \\
Z^{f'}(x,0), & \text{if } S^{f'}(x) \geq \frac{1}{12} \land S^{f'}(x) < \frac{1}{12} \\
-Z^{f'}(x,0), & \text{if } S^{f'}(x) < \frac{1}{12} \land S^{f'}(x) \geq \frac{1}{12} \\
\land Z^{f'}(x,0) < Z^{f'}(x,0) \\
\land Z^{f'}(x,0) \geq Z^{f'}(x,0), 
\end{cases} \]

where \( f'^+ (x, y) = \begin{cases} 
f(x, y), & y \geq 0, \\
1, & y < 0; 
\end{cases} \)
and
\( f^-(x, y) = \begin{cases} 
f(x, -y), & y > 0, \\
1, & y \leq 0; 
\end{cases} \)

remains the class of \( g \) identical to the class of \( Z' \), i.e. \( g \in L_{m+3,p+9} \).

**Theorem 4.3** Let \( f : R^n \to R \) be an \( R \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(3,p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

### 5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the \( \sum^0_n \)–measurable functions and \( \mathbb{R} \)-recursive functions is an open problem.

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**References**