Two hierarchies of $R$-recursive functions

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Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $R$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $N$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called \( \mathbb{R} \)-recursive functions [4].

**Definition 2.1** The set of \( \mathbb{R} \)-recursive functions is generated from the constants 0, 1 by the operations:

1) composition: \( h(\overline{x}) = f(g(\overline{x})) \);

2) differential recursion: \( h(\overline{x}, 0) = f(\overline{x}), \partial_y h(\overline{x}, y) = g(\overline{x}, y, h(\overline{x}, y)) \) (the equivalent formulation can be given by integrals:

\[
h(\overline{x}, y) = f(\overline{x}) + \int_0^y g(\overline{x}, y', h(\overline{x}, y')) \, dy';
\]

3) \( \mu \)-recursion \( h(\overline{x}) = \mu_y f(\overline{x}, y) = \inf \{ y : f(\overline{x}, y) = 0 \} \), where infimum chooses the number \( y \) with the smallest absolute value and for two \( y \) with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if \( h \) is defined by a differential recursion then \( h \) is defined only where a finite and unique solution exists. This is why the set of \( \mathbb{R} \)-recursive functions includes also partial functions. We use (after [4]) the name of \( \mathbb{R} \)-recursive functions in the article, however we should remember that in reality we have partiality here (partial \( \mathbb{R} \)-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive \( y \) or just below some negative \( y \) then the infimum operation returns that \( y \) even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] \( \mu \)-operation is replaced by infinite limits: \( h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x}, y) \), \( h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x}, y) \) then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form \( h(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y) \), which can be in the obvious way obtained from \( \limsup \), \( \liminf \):
**Corollary 2.2** The class of \( R \)-recursive functions is closed under the operations of infinite limits: 
\[
\begin{align*}
  h(\bar{x}) &= \liminf_{y \to \infty} g(\bar{x}, y), \\
  h(\bar{x}) &= \limsup_{y \to \infty} g(\bar{x}, y), \\
  h(\bar{x}) &= \lim_{y \to \infty} g(\bar{x}, y).
\end{align*}
\]

### 3. Hierarchies

The operator \( \mu \) is a key operator in generating the \( R \)-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of \( \mu \) in the definition of a given \( f \).

**Definition 3.1** ([4]) For a given \( R \)-recursive expression \( s(\bar{x}) \), let \( M_{\lambda}(s) \) (the \( \mu \)-number with respect to \( x_i \)) be defined as follows:
\[
\begin{align*}
  M_{\lambda}(0) &= M_{\lambda}(1) = M_{\lambda}(-1) = 0, \\
  M_{\lambda}(f (g_1, g_2, \ldots)) &= \max_j \left( M_{\lambda}(f) + M_{\lambda}(g_j) \right), \\
  M_{\lambda}(h = f + \int_{0}^{\infty} g(\bar{x}, y', h) dy') &= \max \left( M_{\lambda}(f), M_{\lambda}(g), M_{h}(g) \right), \\
  M_{\lambda}(h = f + \int_{0}^{\infty} g(\bar{x}, y', h) dy') &= \max \left( M_{\lambda}(f), M_{\lambda}(g), M_{h}(g) \right), \\
  M_{\lambda}(\mu, f (\bar{x}, y)) &= \max \left( M_{\lambda}(f), M_{\lambda}(f) \right) + 1,
\end{align*}
\]
where \( x \) can be any \( x_1, \ldots, x_n \) for \( \bar{x} = (x_1, \ldots, x_n) \).

For an \( R \)-recursive function \( f \), let \( M(f) = \max_{\lambda}(s) \) minimized over all expressions \( s \) that define \( f \). Now we are ready to define \( M \)-hierarchy (\( \mu \)-hierarchy) as a family of \( M_j = \{ f : M' (f) \leq j \} \).

Let us construct the analogous definition of \( L \)-hierarchy by replacing in the above definition \( M_{\lambda} \) by \( L_{\lambda} \) and changing line (5) to the following form (5'):
\[
\begin{align*}
  L_{\lambda}(\liminf_{y \to \infty} g(\bar{x}, y)) &= L_{\lambda}(\limsup_{y \to \infty} g(\bar{x}, y)) = \\
  &= L_{\lambda}(\lim g(\bar{x}, y)) = \max \left( L_{\lambda}(f), L_{\lambda}(f) \right) + 1. \nonumber
\end{align*}
\]

For an \( R \)-recursive function \( f \), let \( L(f) = \max_{\lambda}, L_{\lambda}(s) \) minimized over all expressions \( S \) that define \( f \) without using the \( \mu \)-operation.
**Definition 3.2** The $\mathbf{L}$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

**Lemma 3.3** The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $\mathbb{R}$-recursive function. After Moore [4] we can conclude that such functions as: $-x, x + y, xy, x/y, e^x, \ln x, y^x, \sin x, \cos x$ are primitive $\mathbb{R}$-recursive.

We can give a few results on some levels of the limit hierarchy.

**Lemma 3.4.** The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

**Proof.** It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases}$ we obtain

$$\text{sgn}(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2 \arctan 1} \text{ and } |x| = \text{sgn}(x)x.$$ 

We should be careful with definitions of functions by cases:

**Lemma 3.5** For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k - 1 \end{cases}$ and $g_i \in L_{n_i}$ for all $1 \leq i \leq k$, $f \in L_m$ the function $h$ belongs to $L_{\max(n_1, \ldots, n_k, m)}$. 
Proof. Let us see that \[ eq(x, y) = \delta(x - y) \in L_1 \] and \[ ge(x, y) = \frac{(\sgn(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1. \] Then of course \[ h(\bar{x}) = \sum_{i=1}^{k-1} g_i(\bar{x}) eq(f(\bar{x}), i - 1) + g_k(\bar{x}) ge(f(\bar{x}), k - 1) \square \]

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 The function \( \Theta(x) \) (equal to 1 if \( x \geq 0 \), otherwise 0), maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n + 1] \) and \( p(x) = 0 \) for \( x \in [2n + 1, 2n + 2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

Proof. We give the proper definitions (from [6]) for these functions. Let \( \Theta(x) = \delta(x - |x|) \), \[ \max(x, y) = x \delta(x - y) + (1 - \delta(x - y))[x \Theta(x - y) + y \Theta(y - x)], \] \[ s(x) = \Theta(\sin(\pi x)). \]

The function \( p(x) \) can be given as \[ s(x) \left( 1 - \delta \left( \frac{\sin \left( \frac{(x-1)\pi}{2} \right)}{2} \right) \right) \], so \( p \in L_2 \).

The floor function we can define by the auxiliary function \( w(0) = 0 \), \( \partial_x w(x) = 2\Theta(-\sin(2\pi x)) \) as
\[ \lfloor x \rfloor = \begin{cases} 2w(x/2) & \text{if } p(x) = 1, \\ 2w((x-1)/2) & \text{if } p(x) = 0. \end{cases} \]

From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \) \( \square \)

Let us recall that if \( f : R^n \rightarrow R \) is an \( R \)-recursive function then the function \( f_{iter}(i, \bar{x}) \) is \( R \)-recursive, too.

Lemma 3.7 Let \( f : R^n \rightarrow R \) belongs to the class \( L_i \), then we have \( f_{iter} : R^{n+1} \rightarrow R \) is in \( L_{\max(2, j)} \).

Proof. The definitions, which were given by Moore [3] \( f_{iter}(i, \bar{x}) = h(2i) \), where \( h(0) = g(0) = \bar{x} \),
\[ \partial_t g(t) = \left[ f(h(t)) - h(t) \right] s(t), \]
\[ \partial_t h(t) = \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)), \]
with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0 \), \( \partial_r t = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( R^l \)-recursive functions \( \gamma_2 : R^2 \rightarrow R, \gamma^1_2, \gamma^2_2 : R \rightarrow R \) such that \((\forall x, y \in R)\gamma^1_2(\gamma_2(x, y)) = x, \ (\forall x, y \in R)\gamma^2_2(\gamma_2(x, y)) = y \), have the following properties: \( \gamma_2, \gamma^1_2 \) are in \( L_{10} \), \( \gamma^2_2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2, \Gamma^1_2, \Gamma^2_2 \), which are coding and decoding functions in the interval \((0,1): \Gamma_2(x, y) = c(x) + c(y)/10\), where
\[ c(x) = \lim_{i \to \infty} z(a(i,x))/10^i + b(i,x)/10^i, \]
and later \( z(x) = \lim_{i \to \infty} z(i,x) \).
\[ z^1_{iter}(i,a_1...a_na_{n+1}...) = a_1...a_n0...a_{n+1}0...a_{n+2}..., \]
\[ b(i,0.a_1a_2...a...) = 0.a_1...a_i \]
\[ (z'(x)) = \begin{cases} 100[x] + 10(x-[x]), & \text{if } [x] \neq x, \\ x, & \text{if } [x] = x, \end{cases} \]
e \( L_4, a, b \in L_4 \). Also \( z_{iter} \) belongs to \( L_4 \), hence \( \Gamma_2(x,y) \in L_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma^1_2(x) \) is in \( L_{10} \) but \( \Gamma^2_2(x) = \Gamma^2_2(10-[10x]) \) so \( \Gamma^2_2 \in L_{14} \).

The functions \( \Gamma_2, \Gamma^1_2, \Gamma^2_2 \) can be extended to all reals by one-to-one \( f:(0,1) \rightarrow R \in L_0 \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \rightarrow R \) and \( \gamma^i_n : R \rightarrow R \) for \( i =1,...,n \) such that
\[(\forall i)(\forall x_1,...,x_n \in R)\gamma^i_n(\gamma_n(x_1,...,x_n)) = x_i \]
in the same class: \( \gamma^i_n \in L_{10} \) and \( (\forall i > 1)\gamma^i_n \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant \( p \in \mathbb{N} \) that for the function

\[
\prod_{z=0}^{y} f(x, z) = \begin{cases} 
  f(x,0) f(x,1) \ldots f(x, \lfloor y - 1 \rfloor), & \text{if } y \geq 1, \\
  1, & \text{if } 0 \leq y < 1, \\
  0, & \text{if } y < 0,
\end{cases}
\]

if the function \( f \) is in the class \( L_m \) then \( \prod_{z=0}^{y} f(x, z) \) is in the class \( L_{m+p} \) (\( p \) is independent of \( m \)).

Proof. By the definitions

\[
t(w) = \gamma_{n+2}(\gamma_{n+2}^{1.n}(w), \gamma_{n+2}^{n+1}(w) + 1, f(\gamma_{n+2}^{1.n}(w), \gamma_{n+2}^{n+1}(w)), \gamma_{n+2}^{n+2}(w))
\]

and

\[
S(x, z) = t_{\lfloor z \rfloor}(s(x, 0) \ldots) = t_{\lfloor z \rfloor}(\lfloor z \rfloor, \gamma_{n+2}(x, 0), 1)
\]

we get the property

\[
\prod_{y=0}^{z} f(x, y) = \gamma_{n+2}^{n+2}(S(x, z)).
\]

From the definition of the limit hierarchy we get \( \prod_{y=0}^{z} f(x, y) \in L_{m+38} \).

In the rest of the paper we will use the constant \( p \) as the number of limits used in the recursive definition of the product \( \prod_{y=0}^{z} f(x, y) \) instead of the value 38. The above constructions are tedious and can be improved with a better approximation of \( p \).

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between \( L \)-hierarchy and \( M \)-hierarchy.

Theorem 4.1 Let \( f : \mathbb{R}^n \to \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then if \( f \in L_i \) then \( f \in M_{10i} \).

Proof. We use a simple induction here. The case \( i = 0 \) is given in Lemma 3.3. Now let us suppose that the thesis is true for \( i = n \). Let \( f \in L_{n+1} \) be defined as

\[
f(x) = \lim_{y \to 0} g(x, y) \quad \text{for} \quad g \in L_n.
\]

Then we can recall Theorem 4.2 from [6] which gives us the following result: to define \( f \) from \( g \) it is necessary to use at
most 10 μ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \). Similar inferences hold for \( \text{lim inf}, \text{lim sup} \). □

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(\vec{x},y): \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) is in the class \( L_m \) then the function \( g: \mathbb{R}^n \rightarrow \mathbb{R}, \quad g(\vec{x}) = \mu_y, f(\vec{x},y) \) is in the class \( L_{m+3p+9} \) is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function \( g: \mathbb{R}^n \rightarrow \mathbb{R}, \quad g(\vec{x}) = \mu_y, f(\vec{x},y) \) for \( f(\vec{x},y): \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) (f - \( \mathbb{R} \)-recursive) replacing the \( \mu \)-operator by limit operation. First we introduced the function

\[
Z_f(\vec{x},z) = \begin{cases} 
\text{undefined} & \text{if } z = 0 \text{ and } \exists y K_f(\vec{x},y) = 0, \\
1 & \text{if } z \neq 0, \\
\text{undefined} & \text{if } z = 0 \text{ and } \forall y K_f(\vec{x},y) \neq 0, \\
\end{cases}
\]

given in the following way:

\[
Z_f(\vec{x},z) = \begin{cases} 
\text{undefined} & \text{if } (z = 0) \land \left( S_f(\vec{x}) < \frac{1}{12} \right), \\
\sqrt{S_f(\vec{x})} - \frac{1}{12}, & \text{if } (z = 0) \land \left( S_f(\vec{x}) \geq \frac{1}{12} \right) \land f(\vec{x},\sqrt{S_f(\vec{x})} - \frac{1}{12}) = 0, \\
-\sqrt{S_f(\vec{x})} - \frac{1}{12}, & \text{if } (z = 0) \land \left( S_f(\vec{x}) \geq \frac{1}{12} \right) \land f(\vec{x},-\sqrt{S_f(\vec{x})} - \frac{1}{12}) = 0, \\
1, & \text{if } z \neq 0.
\end{cases}
\]

where \( S_f(\vec{x}) = \lim_{t \to \infty} S_f(\vec{x},t) + \lim_{t \to -\infty} S_f(\vec{x},t) \). Both functions \( S_1^f, S_2^f \) are defined by an integration

\[
S_i^f(\vec{x},t) = \int y^2 \left( 1 - h_i^f(\vec{x},(-1)^{i+1}y - 1/2,(-1)^{i+1}y + 1/2) \right) dy, \quad i = 1,2
\]

from \( h_i^f(\vec{x},a,b) = \lim_{\epsilon \to 0} \prod_{w=0}^{\epsilon+1} K^f(\vec{x},a + w \frac{b-a}{\epsilon}) \) where \( K^f \) is the characteristic function of \( f \).

Hence we can conclude that if \( K^f \) is in the \( L_s \) then \( Z_f \) is in the class \( L_{s+p+3} \).

Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5]
\[ K^f_\mu(y) = 1 - \lim_{a \to -\infty} \lim_{b \to \infty} \lim_{z \to \infty} G^f(\bar{x}, z, a, b, y), \]

where \( G^f(\bar{x}, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{\lfloor z \rfloor}\) equal subintervals and gives the value 1 for \( y \) from the subintervals, which contains the least zero of \( f \) in \([a, b]\) and value 0 otherwise. Precisely for \( y \) from \([a, a + \frac{b-a}{2^{\lfloor z \rfloor}}]\)

\[
G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f\left(\bar{x}, a, a + \frac{b-a}{2^{\lfloor z \rfloor}}\right) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

for \( y \in \left[a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}}\right] \) (where \( k = 2, 3, ..., 2^n \)) we have:

\[
G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f\left(\bar{x}, a + \frac{(i-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{i(b-a)}{2^{\lfloor z \rfloor}}\right) \neq 0, \\
h^f\left(\bar{x}, a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}}\right) = 0, & \text{otherwise}
\end{cases}
\]

and for \( \bar{Y} \notin [A, B] \) the function \( g^f_\mu \) is equal to 2.

The definition of \( G_f \) is given by the cases with respect to the value of the expression given by \( \prod h^f \), since for \( f \in L_m \), the function \( h_f \in L_{m+p+2} \) and \( G^f \in L_{m+2,p+3} \). Then we have \( K^f_\mu \in L_{m+2,p+6} \). Now we must use the function \( K^f_\mu \) in the same way as \( K^f \) which gives us \( Z_f \) in the class \( L_{m+3,p+9} \). The final definition of \( g(\bar{x}) = \mu, f(\bar{x}, y) \) ([5] Theorem 4.3) given below
\[
g(\bar{x}) = \begin{cases} 
Z^f_+ (\bar{x},0) - Z^f_+ (\bar{x},0), & \text{if } S^f_+ (\bar{x}) < \frac{1}{12} \land S^f_- (\bar{x}) < \frac{1}{12}, \\
Z^f_+ (\bar{x},0), & \text{if } \left( S^f_+ (\bar{x}) \geq \frac{1}{12} \land S^f_- (\bar{x}) < \frac{1}{12} \right) \\
- Z^f_- (\bar{x},0), & \text{if } \left( S^f_+ (\bar{x}) < \frac{1}{12} \land S^f_- (\bar{x}) \geq \frac{1}{12} \right) \\
& \text{or } \left( S^f_+ (\bar{x}) < \frac{1}{12} \land S^f_- (\bar{x}) < \frac{1}{12} \land Z^f_+ (\bar{x},0) < Z^f_- (\bar{x},0) \right), \\
& \text{or } \left( S^f_+ (\bar{x}) < \frac{1}{12} \land S^f_- (\bar{x}) < \frac{1}{12} \land Z^f_+ (\bar{x},0) \geq Z^f_- (\bar{x},0) \right), 
\end{cases}
\]

where

\[
f^+ (\bar{x},y) = \begin{cases} 
f (\bar{x},y), & y \geq 0, \\
1, & y < 0; 
\end{cases}
\]

\[
f^- (\bar{x},y) = \begin{cases} 
f (\bar{x},-y), & y > 0, \\
1, & y \leq 0; 
\end{cases}
\]

remains the class of \( g \) identical to the class of \( Z^f \), i.e. \( g \in L_{m+3,p+9} \).

**Theorem 4.3** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(i,p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

### 5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum_0^n$–measurable functions and $R$-recursive functions is an open problem.

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References