Two hierarchies of $R$-recursive functions

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Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also of interest to mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $R$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $\mathbb{N}$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called $\mathbb{R}$-recursive functions [4].

**Definition 2.1** The set of $\mathbb{R}$-recursive functions is generated from the constants $0, 1$ by the operations:

1) composition: $h(\vec{x}) = f(g(\vec{x}))$;

2) differential recursion: $h(\vec{x}, 0) = f(\vec{x}), \partial_y h(\vec{x}, y) = g(\vec{x}, y, h(\vec{x}, y))$ (the equivalent formulation can be given by integrals:

$h(\vec{x}, y) = f(\vec{x}) + \int_0^y g(\vec{x}, y', h(\vec{x}, y')) dy'$;

3) $\mu$-recursion $h(\vec{x}) = \mu_y f(\vec{x}, y) = \inf \{y : f(\vec{x}, y) = 0\}$, where infimum chooses the number $y$ with the smallest absolute value and for two $y$ with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if $h$ is defined by a differential recursion then $h$ is defined only where a finite and unique solution exists. This is why the set of $\mathbb{R}$-recursive functions includes also partial functions. We use (after [4]) the name of $\mathbb{R}$-recursive functions in the article, however we should remember that in reality we have partiality here (partial $\mathbb{R}$-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive $y$ or just below some negative $y$ then the infimum operation returns that $y$ even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] $\mu$-operation is replaced by infinite limits: $h(\vec{x}) = \liminf_{y \to \infty} g(\vec{x}, y), h(\vec{x}) = \limsup_{y \to \infty} g(\vec{x}, y)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(\vec{x}) = \lim_{y \to \infty} g(\vec{x}, y)$, which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of $R$-recursive functions is closed under the operations of infinite limits: $h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x}, y)$, $h(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$.

3. Hierarchies

The operator $\mu$ is a key operator in generating the $R$-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of $\mu$ in the definition of a given $f$.

Definition 3.1 ([4]) For a given $R$-recursive expression $s(\overline{x})$, let $M_s(s)$ (the $\mu$-number with respect to $x_i$) be defined as follows:

$$M_s(0) = M_s(1) = M_s(-1) = 0,$$

$$M_s\left(f\left(g_1, g_2, \ldots\right)\right) = \max_j \left(M_{s_j}(f) + M_s(g_j)\right),$$

$$M_s\left(h = f + \int_0^\gamma g(\overline{x}, y', h) dy'\right) = \max\left(M_s(f), M_s(g), M_h(g)\right),$$

$$M_s\left(h = f + \int_0^\gamma g(\overline{x}, y', h) dy'\right) = \max\left(M_y(g), h(g)\right),$$

$$M_s\left(\mu, f(\overline{x}, y)\right) = \max\left(M_s(f), M_y(f)\right) + 1,$$

where $x$ can be any $x_1, \ldots, x_n$ for $\overline{x} = (x_1, \ldots, x_n)$.

For an $R$-recursive function $f$, let $M(f) = \max_{s_i}(s)$ minimized over all expressions $s$ that define $f$. Now we are ready to define $M$-hierarchy ($\mu$-hierarchy) as a family of $M_j = \{ f : M'(f) \leq j \}$.

Let us construct the analogous definition of $L$-hierarchy by replacing in the above definition $M_s$ by $L_s$ and changing line (5) to the following form (5'):

$$L_s\left(\liminf_{y \to \infty} g(\overline{x}, y)\right) = L_s\left(\limsup_{y \to \infty} g(\overline{x}, y)\right) =$$

$$= L_s\left(\lim g(\overline{x}, y)\right) = \max\left(L_s(f), L_y(f)\right) + 1.$$

For an $R$-recursive function $f$, let $L(f) = \max_{s_i}(s)$ minimized over all expressions $S$ that define $f$ without using the $\mu$-operation.
**Definition 3.2** The $L$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$. 

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

**Lemma 3.3** The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $R$-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, $xy$, $x/y$, $e^x$, $\ln x$, $y^x$, $\sin x$, $\cos x$ are primitive $R$-recursive.

We can give a few results on some levels of the limit hierarchy.

**Lemma 3.4.** The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

**Proof.** It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases}$ we obtain $\text{sgn}(x) = \liminf_{y \to \infty} \arctan xy / 2 \arctan 1$ and $|x| = \text{sgn}(x)x$.

We should be careful with definitions of functions by cases:

**Lemma 3.5** For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \ldots & \ldots, \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k - 1 \end{cases}$ and $g_i \in L_{m_i}$ for all $1 \leq i \leq k$, $f \in L_m$, the function $h$ belongs to $L_{\max(m_1, \ldots, m_k, m+1)}$.
Proof. Let us see that \( eq(x, y) = \delta (x - y) \in L_1 \) and \( ge(x, y) = \frac{(\text{sgn}(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1 \). Then of course
\[
h(x) = \sum_{i=1}^{k} g_i(x) eq(f(x), i-1) + g_k(x) ge(f(x), k-1)
\]
\( \square \)

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 The function \( \Theta(x) \) (equal to 1 if \( x \geq 0 \), otherwise 0), maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n+1] \) and \( p(x) = 0 \) for \( x \in [2n+1, 2n+2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

Proof. We give the proper definitions (from [6]) for these functions. Let
\[
\Theta(x) = \delta (x - |x|),
\]
\[
\max(x, y) = x\delta (x - y) + (1 - \delta (x - y))[x\Theta(x - y) + y\Theta(y - x)],
\]
\[
s(x) = \Theta(\sin(\pi x)).
\]

The function \( p(x) \) can be given as \( s(x) \left( 1 - \delta \left( \frac{\sin(x-1/\pi)}{2} \right) \right) \), so \( p \in L_2 \).

The floor function we can define by the auxiliary function \( w(0) = 0, \partial_w x = 2\Theta(-\sin(2\pi x)) \) as
\[
\lfloor x \rfloor = \begin{cases} 2w(x/2) & \text{if } p(x) = 1, \\
2w((x-1)/2) & \text{if } p(x) = 0. \end{cases}
\]

From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \) \( \square \)

Let us recall that if \( f : R^n \rightarrow R \) is an \( R \)-recursive function then the function \( f_{iter}(i, \bar{x}) \) is \( R \)-recursive, too.

Lemma 3.7 Let \( f : R^n \rightarrow R \) belongs to the class \( L_4 \), then we have \( f_{iter} : R^{n+1} \rightarrow R \) is in \( L_{\max(2, j)} \).

Proof. The definitions, which were given by Moore [3] \( f_{iter}(i, \bar{x}) = h(2i) \), where
\[
h(0) = g(0) = \bar{x},
\]
\[ \partial, g(t) = \left[ f(h(t)) - h(t) \right] s(t), \]
\[ \partial, h(t) = \frac{g(t) - h(t)}{r(t)} (1 - s(t)), \]

with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0 \), \( \partial, r(t) = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( R^l \)-recursive functions \( \gamma_2 : R^2 \to R \), \( \gamma_1^l, \gamma_2^l : R \to R \) such that \((\forall x, y \in R)\gamma_2^l(\gamma_2(x, y)) = x \), \((\forall x, y \in R)\gamma_1^l(\gamma_2(x, y)) = y \), have the following properties: \( \gamma_2 \), \( \gamma_1^l \) are in \( L_{10} \), \( \gamma_2^l \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2 \), \( \Gamma_1^l \), \( \Gamma_2^l \), which are coding and decoding functions in the interval \((0,1) : \Gamma_2(x, y) = c(x) + c(y)/10\), where

\[ c(x) = \lim_{i \to \infty} z(a(i, x))\big/10^i + b(i, x)/10', \]

and later \( z(x) = \lim_{i \to \infty} z_{iter}^l(i, x) \).

\[ z_{iter}^l(i, a_1...a_n a_{n+1}...) = a_1...a_n 0...0 a_{n+1}...a_i, \]
\[ b(i, 0.a_1 a_2...a_i...) = 0.a_1...a_i, \]
\[ b(i, 0.a_1 a_2...a_i...) = 0.0\mathbf{\xi}a_{i+1}... \]

\[ (z'(x)) = \begin{cases} 100[0.1x] + 10(x - \lfloor x \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x; \end{cases} \]

Also \( z_{iter}^l \) belongs to \( L_4 \), hence \( \Gamma_2(x, y) \in L_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma_1^l(x) \) is in \( L_{10} \), but \( \Gamma_2^l(x) = \Gamma_1^l (10 - \lfloor 10x \rfloor) \) so \( \Gamma_2^l \in L_{14} \).

The functions \( \Gamma_2 \), \( \Gamma_1^l \), \( \Gamma_2^l \) can be extended to all reals by one-to-one \( f:(0,1) \to R \in L_0 \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \to R \) and \( \gamma_1^l : R \to R \) for \( i = 1,\ldots,n \) such that

\[ (\forall i)(\forall x_1,...,x_n \in R)\gamma_1^l(\gamma_n(x_1,...,x_n)) = x_i \]

in the same class: \( \gamma_n, \gamma_1^l \in L_{10} \) and \( (\forall i \geq 1)\gamma_i^l \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant $p \in \mathbb{N}$ that for the function

$$\prod_{y=0}^{y} f(x, y) = \begin{cases} f(x, 0) f(x, 1) \cdots f(x, y-1), & \text{if } y \geq 1, \\ 1, & \text{if } 0 \leq y < 1, \\ 0, & \text{if } y < 0, \end{cases}$$

if the function $f$ is in the class $L_m$ then $\prod_{y=0}^{y} f(x, y)$ is in the class $L_{m+p}$ ($p$ is independent of $m$).

Proof. By the definitions

$$t(w) = \gamma_{n+2} \left( \gamma_{n+2} \left( w \right), \gamma_{n+2} \left( w \right) + 1, f \left( \gamma_{n+2} \left( w \right), \gamma_{n+2} \left( w \right) \right) \right)$$

and

$$S(x, z) = t \left( x, t \left( s(0), \ldots, s(0) \right) \right) = t_{i+1} \left( \left[ \left[ z \right], \gamma_{n+2} \left( x, 0, 1 \right) \right] \right)$$

we get the property

$$\prod_{y=0}^{y} f(x, y) = \gamma_{n+2} \left( S(x, z) \right).$$

From the definition of the limit hierarchy we get $\prod_{y=0}^{y} f(x, y) \in L_{m+38} \square$

In the rest of the paper we will use the constant $p$ as the number of limits used in the recursive definition of the product $\prod_{y=0}^{y} f(x, y)$ instead of the value 38.

The above constructions are tedious and can be improved with a better approximation of $p$.

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between $L$-hierarchy and $M$-hierarchy.

Theorem 4.1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $\mathbb{R}$-recursive function. Then if $f \in L_i$ then $f \in M_{10i}$.

Proof. We use a simple induction here. The case $i = 0$ is given in Lemma 3.3. Now let us suppose that the thesis is true for $i = n$. Let $f \in L_{n+1}$ be defined as $f(x) = \lim_{y \to \infty} g(x, y)$ for $g \in L_n$. Then we can recall Theorem 4.2 from [6] which gives us the following result: to define $f$ from $g$ it is necessary to use at
most $10$ $\mu$-operation. Hence for $g \in M_{10n}$ the function $f$ satisfies $f \in M_{10n+10}$. Similar inferences hold for $\lim \inf, \lim \sup$. □

Now we can give the result about the 'limit complexity' of the infimum operator $\mu$.

**Lemma 4.2** If $f(x,y): R^{n+1} \rightarrow R$ is in the class $L_m$ then the function $g: R^n \rightarrow R$, $g(x) = \mu_y f(x,y)$ is in the class $L_{m+3p+9}$ is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function $g: R^n \rightarrow R$, $g(x) = \mu_y f(x,y)$ for $f(x,y): R^{n+1} \rightarrow R$ ($f$ - $R$-recursive) replacing the $\mu$-operator by limit operation. First we introduced the function

$$Z'(x,z) = \begin{cases} \inf_y \{f : K'(x,y) = 0\}, & \text{if } z = 0 \text{ and } \exists y K'(x,y) = 0, \\ \text{undefined} & \text{if } z = 0 \text{ and } \forall y K'(x,y) \neq 0, \\ 1 & \text{if } z \neq 0, \end{cases}$$

given in the following way:

$$Z'(x,z) = \begin{cases} \text{undefined} & \text{if } (z = 0) \land \left(S'(x) < \frac{1}{12}\right), \\ \sqrt{S'(x) - \frac{1}{12}}, & \text{if } (z = 0) \land \left(S'(x) \geq \frac{1}{12}\right) \land f(x,\sqrt{S'(x)} - \frac{1}{12}) = 0, \\ -\sqrt{S'(x) - \frac{1}{12}}, & \text{if } (z = 0) \land \left(S'(x) \geq \frac{1}{12}\right) \land f(x,-\sqrt{S'(x)} - \frac{1}{12}) = 0, \\ 1, & \text{if } z \neq 0, \end{cases}$$

where $S'(x) = \lim_{t \rightarrow \infty} S_i'(x,t) + \lim_{t \rightarrow \infty} S_2'(x,t)$. Both functions $S_1', S_2'$ are defined by an integration

$$S_i'(x,t) = \int y^2 \left(1 - h'(x,-1)^{i+1} y - 1/2,(-1)^{i+1} y + 1/2\right) dy, \ i = 1,2$$

from $h'(x,a,b) = \lim \inf_{t \rightarrow \infty} \prod_{w=0}^{\infty} K'(x,a + w\frac{b-a}{z})$ where $K'$ is the characteristic function of $f$.

Hence we can conclude that if $K'$ is in the $L_s$ then $Z_f$ is in the class $L_{s+p+3}$. Let us finish with the definition of the characteristic function of the infimum of zeros of $f$ (see Theorem 4.2 from [5]
\[ K^f_\mu(y) = 1 - \lim_{a \to -\infty} \lim_{b \to \infty} \lim_{z \to \infty} G^f(\bar{x}, z, a, b, y), \]

where \( G^f(\bar{x}, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{|z|}\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a, b]\) and value 0 otherwise. Precisely for \(y\) from \([a, a + \frac{b-a}{2^{|z|}}]\)

\[ G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f\left(\bar{x}, a, a + \frac{b-a}{2^{|z|}}\right) = 0, \\
0, & \text{otherwise}
\end{cases} \]

for \(y \in \left( a + \frac{(k-1)(b-a)}{2^{|z|}}, a + \frac{k(b-a)}{2^{|z|}} \right)\) (where \(k = 2, 3, ..., 2^n\)) we have:

\[ G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f\left(\bar{x}, a + \frac{(i-1)(b-a)}{2^{|z|}}, a + \frac{i(b-a)}{2^{|z|}}\right) \neq 0, \\
\wedge h^f\left(\bar{x}, a + \frac{(k-1)(b-a)}{2^{|z|}}, a + \frac{k(b-a)}{2^{|z|}}\right) = 0, & \text{otherwise}
\end{cases} \]

and for \(Y \notin [A, B]\) the function \(g_x^f\) is equal to 2.

The definition of \(G_f\) is given by the cases with respect to the value of the expression given by \(\prod h^f\), since for \(f \in L_m\), the function \(h_f \in L_{m+p+2}\) and \(G_f \in L_{m+2+p+3}\). Then we have \(K^f_\mu \in L_{m+2+p+6}\). Now we must use the function \(K^f_\mu\) in the same way as \(K^f\) which gives us \(Z_f\) in the class \(L_{m+3+p+9}\). The final definition of \(g(\bar{x}) = \mu_y f(\bar{x}, y)\) ([5] Theorem 4.3) given below
\[
g(\bar{x}) = \begin{cases} 
Z^f(\bar{x}, 0) - Z^f(\bar{x}, 0), & \text{if } S^f(\bar{x}) < \frac{1}{12} \land S^f(\bar{x}) < \frac{1}{12}, \\
Z^f(\bar{x}, 0), & \text{if } \left( S^f(\bar{x}) \geq \frac{1}{12} \land S^f(\bar{x}) < \frac{1}{12} \right) \\
-Z^f(\bar{x}, 0), & \text{or } \left( S^f(\bar{x}) < \frac{1}{12} \land S^f(\bar{x}) \geq \frac{1}{12} \right) \\
-Z^f(\bar{x}, 0) \land Z^f(\bar{x}, 0) < Z^f(\bar{x}, 0) \right) , & \text{or } \left( S^f(\bar{x}) < \frac{1}{12} \land S^f(\bar{x}) < \frac{1}{12} \right) \\
-Z^f(\bar{x}, 0) \land Z^f(\bar{x}, 0) \geq Z^f(\bar{x}, 0) \right) , & \text{or } \left( S^f(\bar{x}) < \frac{1}{12} \land S^f(\bar{x}) \geq \frac{1}{12} \right) 
\end{cases}
\]

where
\[
f^+(\bar{x}, y) = \begin{cases} 
 f(\bar{x}, y), & y \geq 0, \\
1, & y < 0; 
\end{cases} \quad f^-(\bar{x}, y) = \begin{cases} 
f(\bar{x}, -y), & y > 0, \\
1, & y \leq 0; 
\end{cases}
\]
remains the class of \( g \) identical to the class of \( Z^f \), i.e. \( g \in L_{m+3p+q} \). □

**Theorem 4.3** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be an \( \mathbb{R} \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(3p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

### 5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes...
[7]. Also the kind of a connection between the $\sum^0_n$-measurable functions and R-recursive functions is an open problem.

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References