Two hierarchies of $R$-recursive functions

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Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $R$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $\mathbb{N}$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called $R$-recursive functions [4].

**Definition 2.1** The set of $R$-recursive functions is generated from the constants 0,1 by the operations:

1) composition: $h(x) = f(g(x))$;

2) differential recursion: $h(x,0) = f(x), h(x,y) = g(x,y,h(x,y))$ (the equivalent formulation can be given by integrals: $h(x,y) = f(x) + \int_0^y g(x,y',h(x,y')) dy'$);

3) $\mu$-recursion $h(x) = \mu_y f(x,y) = \inf \{y: f(x,y) = 0\}$, where infimum chooses the number $y$ with the smallest absolute value and for two $y$ with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if $h$ is defined by a differential recursion then $h$ is defined only where a finite and unique solution exists. This is why the set of $R$-recursive functions includes also partial functions. We use (after [4]) the name of $R$-recursive functions in the article, however we should remember that in reality we have partiality here (partial $R$-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive $y$ or just below some negative $y$ then the infimum operation returns that $y$ even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] $\mu$-operation is replaced by infinite limits: $h(x) = \liminf_{y \to \infty} g(x,y)$, $h(x) = \limsup_{y \to \infty} g(x,y)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(x) = \lim_{y \to \infty} g(x,y)$, which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of \( R \)-recursive functions is closed under the operations of infinite limits: 
\[
\begin{align*}
\liminf_{y \to \infty} g(x,y) &= h(x), \\
\limsup_{y \to \infty} g(x,y) &= h(x), \\
\lim_{y \to \infty} g(x,y) &= h(x).
\end{align*}
\]

3. Hierarchies

The operator \( \mu \) is a key operator in generating the \( R \)-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of \( \mu \) in the definition of a given \( f \).

Definition 3.1 ([4]) For a given \( R \)-recursive expression \( s(x) \), let \( M_x(s) \) (the \( \mu \)-number with respect to \( x_i \)) be defined as follows:
\[
M_x(0) = M_x(1) = M_x(-1) = 0, \\
M_x\left(f\left(g_1, g_2, \ldots \right)\right) = \max_j \left( M_{x_j}\left(f\right) + M_{x_j}\left(g_j\right) \right), \\
M_x\left(h = f + \int_0^\infty g(x, y', h)dy' \right) = \max \left( M_x\left(f\right), M_x\left(g\right), M_h\left(g\right) \right), \\
M_y\left(h = f + \int_0^\infty g(x, y', h)dy' \right) = \max \left( M_y\left(g\right), M_h\left(g\right) \right), \\
M_x\left(\mu_x, f\left(x, y\right)\right) = \max \left( M_x\left(f\right), M_y\left(f\right) \right) + 1,
\]
where \( x \) can be any \( x_1, \ldots, x_n \) for \( \overline{x} = (x_1, \ldots, x_n) \).

For an \( R \)-recursive function \( f \), let \( M(f) = \max_{s} M_x(s) \) minimized over all expressions \( s \) that define \( f \). Now we are ready to define \( M \)-hierarchy (\( \mu \)-hierarchy) as a family of \( M_j = \{ f : M'(f) \leq j \} \).

Let us construct the analogous definition of \( L \)-hierarchy by replacing in the above definition \( M_x \) by \( L_x \) and changing line (5) to the following form (5'):
\[
L_x\left(\liminf_{y \to \infty} g(x, y)\right) = \max \left( L_x\left(f\right), L_y\left(f\right) \right) + 1.
\]

For an \( R \)-recursive function \( f \), let \( L(f) = \max_{s} L_{\xi}(s) \) minimized over all expressions \( S \) that define \( f \) without using the \( \mu \)-operation.
**Definition 3.2** The $L$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

**Lemma 3.3** The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $R$-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, $xy$, $x/y$, $e^x$, $\ln x$, $y^x$, $\sin x$, $\cos x$ are primitive $R$-recursive.

We can give a few results on some levels of the limit hierarchy.

**Lemma 3.4.** The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

**Proof.** It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \lim \inf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\lim \inf_{y \to \infty} \arctan xy = \begin{cases} \pi / 2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi / 2, & \text{if } x < 0, \end{cases}$ we obtain $\sgn(x) = \frac{\lim \inf_{y \to \infty} \arctan xy}{2 \arctan 1}$ and $|x| = \sgn(x)x$.

We should be careful with definitions of functions by cases:

**Lemma 3.5** For $h(\overline{x}) = \begin{cases} g_1(\overline{x}), & \text{if } f(\overline{x}) = 0, \\ g_2(\overline{x}), & \text{if } f(\overline{x}) = 1, \\ \vdots, & \text{if } f(\overline{x}) \geq k - 1, \end{cases}$ and $g_i \in L_{n_i}$ for all $1 \leq i \leq k$, $f \in L_M$ the function $h$ belongs to $L_{\max\{n_1,\ldots,n_k,M+1\}}$. 

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Proof. Let us see that $eq(x, y) = \delta(x - y) \in L_1$ and

$$ge(x, y) = \frac{(\text{sgn}(x - y) + eq(x, y))}{2} + \frac{1}{2} \in L_1.$$ 

Then of course

$$h(\bar{x}) = \sum_{i=1}^{k-1} g_i(\bar{x})eq(f(\bar{x}), i-1) + g_k(\bar{x})ge(f(\bar{x}), k-1) \square$$

Of course this result can be easily extended to other forms of definitions by cases.

**Lemma 3.6** The function $\Theta(x)$ (equal to 1 if $x \geq 0$, otherwise 0), maximum $\max(x, y)$, square-wave function $s$ are in $L_2$, the function $p(x)$ such that $p(x) = 1$ for $x \in [2n, 2n+1]$ and $p(x) = 0$ for $x \in [2n+1, 2n+2]$ is in $L_2$ and the floor function $\lfloor x \rfloor$ is in $L_3$.

**Proof.** We give the proper definitions (from [6]) for these functions. Let

$$\Theta(x) = \delta(x - |x|),$$

$$\max(x, y) = x\delta(x - y) + (1 - \delta(x - y))\left[x\Theta(x - y) + y\Theta(y - x)\right],$$

$$s(x) = \Theta(\sin(\pi x)).$$

The function $p(x)$ can be given as $s(x)\left(1 - \delta\left(\sin\left(\frac{x-1}{2}\pi\right)\right)\right)$, so $p \in L_2$.

The floor function we can define by the auxiliary function $w(0) = 0$, $\partial_x w(x) = 2\Theta(-\sin(\pi x))$ as

$$\lfloor x \rfloor = \begin{cases} 2w(x/2) & \text{if } p(x) = 1, \\ 2w((x-1)/2) & \text{if } p(x) = 0. \end{cases}$$

From the above equation we have $\lfloor x \rfloor$ in $L_3$. \square

Let us recall that if $f : R^n \rightarrow R$ is an $R$-recursive function then the function $f_{\text{iter}}(i, \bar{x})$ is $R$-recursive, too.

**Lemma 3.7** Let $f : R^n \rightarrow R$ belongs to the class $L_i$, then we have $f_{\text{iter}} : R^{n+1} \rightarrow R$ is in $L_{\max(2, j)}$.

**Proof.** The definitions, which were given by Moore [3] $f_{\text{iter}}(i, \bar{x}) = h(2i)$, where $h(0) = g(0) = \bar{x}$,
with \( s \) - a square wave function in \( L_2 \) and \( r(0)=0 \), \( \partial h(t) = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( \mathbb{R}^l \)-recursive functions \( \gamma_2 : \mathbb{R}^2 \to \mathbb{R}, \gamma_2, \gamma_2' : \mathbb{R} \to \mathbb{R} \) such that \((\forall x, y \in \mathbb{R})\gamma_2'(\gamma_2(x, y)) = x \), \((\forall x, y \in \mathbb{R})\gamma_2'(\gamma_2(x, y)) = y \), have the following properties: \( \gamma_2, \gamma_2' \) are in \( L_{10} \), \( \gamma_2^2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2, \Gamma_2', \Gamma_2^2 \), which are coding and decoding functions in the interval \((0,1) : \Gamma_2(x, y) = c(x) + c(y)/10 \), where

\[
c(x) = \lim_{i \to \infty} z_i(a(i, x)) = 10^{2i} + b(i, x)/10',
\]

and later \( z_i = \lim_{i \to \infty} z_{iter}(i, x) \),

\[
z_{iter}(i, a_1...a_n, a_{n+1}...) = a_1...a_n0...a_{n+1}0...a_{n+2}...,
\]

\[
a(i, 0.a_1a_2...a_i...) = 0.a_1...a_i,
\]

\[
b(i, 0.a_1a_2...a_i...) = 0.a_1...a_i0...a_{i+1},
\]

\[
(z'(x) = \begin{cases} 100\lfloor x \rfloor + 10(x - \lfloor x \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x, \end{cases} \in \mathbb{L}_4, a, b \in \mathbb{L}_4. \]

Also \( z_{iter} \) belongs to \( \mathbb{L}_4 \), hence \( \Gamma_2(x, y) \in \mathbb{L}_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma_2'(x) \) is in \( \mathbb{L}_{10} \), but \( \Gamma_2''(x) = \Gamma_2'(10 - \lfloor 10x \rfloor) \) so \( \Gamma_2'' \in \mathbb{L}_{14} \).

The functions \( \Gamma_2, \Gamma_2', \Gamma_2^2 \) can be extended to all reals by one-to-one \( f : (0,1) \to \mathbb{R} \in \mathbb{L}_0 \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : \mathbb{R}^n \to \mathbb{R} \) and \( \gamma_n' : \mathbb{R} \to \mathbb{R} \) for \( i = 1, ..., n \) such that

\[
(\forall i)(\forall x_1, ..., x_n \in \mathbb{R})\gamma_n'(\gamma_n(x_1, ..., x_n)) = x_i
\]

in the same class: \( \gamma_n, \gamma_n' \in \mathbb{L}_{10} \) and \((\forall i > 1)\gamma_n' \in \mathbb{L}_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values in some integer points.
Lemma 3.9 There exists such constant $p \in \mathbb{N}$ that for the function

$$\prod_{z=0}^{y} f(x, z) = \begin{cases} f(x, 0) f(x, 1) \ldots f(x, y-1), & \text{if } y \geq 1, \\ 1, & \text{if } 0 \leq y < 1, \\ 0, & \text{if } y < 0, \end{cases}$$

if the function $f$ is in the class $L_m$ then $\prod_{z=0}^{y} f(x, z)$ is in the class $L_{m+p}$ ($p$ is independent of $m$).

Proof. By the definitions

$$t(w) = \gamma_{n+2}^{1,n} (w) \cdot \gamma_{n+2}^{n+1} (w) + 1, f \left( \gamma_{n+2}^{1,n} (w), \gamma_{n+2}^{n+1} (w) \right)$$

and

$$S(x, z) = t \left( \left[ t \left( \left[ z \right] \right) \right] \right)$$

we get the property

$$\prod_{y=0}^{z} f(x, y) = \gamma_{n+2}^{n+2} (S(x, z)).$$

From the definition of the limit hierarchy we get $\prod_{y=0}^{z} f(x, y) \in L_{m+38}$.

In the rest of the paper we will use the constant $p$ as the number of limits used in the recursive definition of the product $\prod_{y=0}^{z} f(x, y)$ instead of the value 38. The above constructions are tedious and can be improved with a better approximation of $p$.

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between $L$-hierarchy and $M$-hierarchy.

Theorem 4.1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $\mathbb{R}$-recursive function. Then if $f \in L_i$ then $f \in M_{10i}$.

Proof. We use a simple induction here. The case $i = 0$ is given in Lemma 3.3. Now let us suppose that the thesis is true for $i = n$. Let $f \in L_{n+1}$ be defined as $f(x) = \lim_{y \to \infty} g(x, y)$ for $g \in L_n$. Then we can recall Theorem 4.2 from [6] which gives us the following result: to define f from g it is necessary to use at
most 10 μ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \). Similar inferences hold for \( \lim \inf \), \( \lim \sup \).

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(\bar{x},y) : R^{n+1} \to R \) is in the class \( L_m \) then the function \( g : R^n \to R \), \( g(\bar{x}) = \mu_y f(\bar{x},y) \) is in the class \( L_{m+3p+9} \) is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function \( g : R^n \to R \), \( g(\bar{x}) = \mu_y f(\bar{x},y) \) for \( f(\bar{x},y) : R^{n+1} \to R \) (\( f \) - \( R \)-recursive) replacing the \( \mu \)-operator by limit operation. First we introduced the function

\[
Z_f(\bar{x},z) = \begin{cases} 
\inf_y \left\{ f : K'(\bar{x}, y) = 0 \right\}, & \text{if } z = 0 \text{ and } \exists y K'(\bar{x}, y) = 0, \\
\text{undefined} & \text{if } z = 0 \text{ and } \forall y K'(\bar{x}, y) \neq 0, \\
1 & \text{if } z \neq 0,
\end{cases}
\]

given in the following way:

\[
Z_f(\bar{x},z) = \begin{cases} 
\text{undefined} & \text{if } (z = 0) \land \left( S_f(\bar{x}) < \frac{1}{12} \right) , \\
\sqrt{S_f(\bar{x}) - \frac{1}{12}}, & \text{if } (z = 0) \land \left( S_f(\bar{x}) \geq \frac{1}{12} \right) \land f(\bar{x}, \sqrt{S_f(\bar{x}) - \frac{1}{12}}) = 0, \\
-\sqrt{S_f(\bar{x}) - \frac{1}{12}}, & \text{if } (z = 0) \land \left( S_f(\bar{x}) \geq \frac{1}{12} \right) \land f(\bar{x}, -\sqrt{S_f(\bar{x}) - \frac{1}{12}}) = 0, \\
1, & \text{if } z \neq 0.
\end{cases}
\]

where \( S_f(\bar{x}) = \lim_{t \to \infty} S_f(\bar{x},t) + \lim_{t \to \infty} S_f(\bar{x},t) \). Both functions \( S_1, S_2 \) are defined by an integration

\[
S_i(\bar{x},t) = \int y^2 \left( 1 - h^i(f(\bar{x},(-1)^{i+1}y - 1/2,(-1)^{i+1}y + 1/2)) \right) dy, \quad i = 1,2
\]

from \( h^i(f(\bar{x},a,b)) = \liminf_{y \to 0} \prod_{w=0}^{\infty} K'(\bar{x},a + w \frac{b-a}{\bar{z}}) \) where \( K' \) is the characteristic function of \( f \).

Hence we can conclude that if \( K' \) is in the \( L_s \), then \( Z_f \) is in the class \( L_{s+p+3} \).

Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5].
\[ K^f_{\mu}(y) = 1 - \lim_{n \to \infty} \lim_{b \to \infty} \lim_{z \to \infty} G^f(\bar{x}, z, a, b, y), \]

where \( G^f(\bar{x}, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{[z]}\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a, b]\) and value 0 otherwise. Precisely for \(y\) from \([a, a + \frac{b-a}{2^{[z]}}]\)

\[
G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f(\bar{x}, a, a + \frac{b-a}{2^{[z]}}, a + \frac{b-a}{2^{[z]}}) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

for \(y \in \left(a + \frac{(k-1)(b-a)}{2^{[z]}} + a + \frac{k(b-a)}{2^{[z]}}\right)\) (where \(k = 2, 3, \ldots, 2^n\)) we have:

\[
G^f(\bar{x}, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f(\bar{x}, a + \frac{(i-1)(b-a)}{2^{[z]}}, a + \frac{i(b-a)}{2^{[z]}}) \neq 0 \\
\land h^f(\bar{x}, a + \frac{(k-1)(b-a)}{2^{[z]}}, a + \frac{k(b-a)}{2^{[z]}}) = 0, & \text{otherwise}
\end{cases}
\]

and for \(Y \not\in [A, B]\) the function \(g^f_x\) is equal to 2.

The definition of \(G^f\) is given by the cases with respect to the value of the expression given by \(\prod h^f\), since for \(f \in L_m\), the function \(h_f \in L_{m+p+2}\) and \(G^f \in L_{m+2+p+3}\). Then we have \(K^f_{\mu} \in L_{m+2+p+6}\). Now we must use the function \(K^f_{\mu}\) in the same way as \(K^f\) which gives us \(Z_f\) in the class \(L_{m+3+p+9}\). The final definition of \(g(\bar{x}) = \mu, f(\bar{x}, y)\) ([5] Theorem 4.3) given below.
\[ g(\overline{x}) = \begin{cases} 
Z^{f^+}(\overline{x},0) - Z^{f^-}(\overline{x},0), & \text{if } S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12}, \\
Z^{f^+}(\overline{x},0), & \text{if } \left( S^{f^+}(\overline{x}) \geq \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12} \right) \lor \left( S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12} \land Z^{f^+}(\overline{x},0) < Z^{f^-}(\overline{x},0) \right), \\
-Z^{f^-}(\overline{x},0), & \text{if } \left( S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) \geq \frac{1}{12} \right) \lor \left( S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12} \land Z^{f^+}(\overline{x},0) \geq Z^{f^-}(\overline{x},0) \right), 
\end{cases} \]

where \( f^+(\overline{x}, y) = \begin{cases} f(\overline{x}, y), & y \geq 0, \\
1, & y < 0; \end{cases} \)
\( f^-(\overline{x}, y) = \begin{cases} f(\overline{x}, -y), & y > 0, \\
1, & y \leq 0; \end{cases} \)

remains the class of \( g \) identical to the class of \( Z^f \), i.e. \( g \in L_{m+3,p+9} \).

**Theorem 4.3** Let \( f : R^n \to R \) be an \( R \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(i,p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

**5. Conclusions**

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum^0_m$-measurable functions and \( R \)-recursive functions is an open problem.

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References