Two hierarchies of $R$-recursive functions

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Abstract

In the paper some aspects of complexity of $R$-recursive functions are considered. The limit hierarchy of $R$-recursive functions is introduced by the analogy to the $\mu$-hierarchy. Then its properties and relations to the $\mu$-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals $R$ (called $R$-recursive functions) in the analogous way to the classical recursive functions on the natural numbers $\mathbb{N}$. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation $\mu$, which is used to construct $\mu$-hierarchy of $R$-recursive functions.

It was shown [5] that the zero-finding operator $\mu$ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to $\mu$-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called \( R \)-recursive functions [4].

**Definition 2.1** The set of \( R \)-recursive functions is generated from the constants \( 0,1 \) by the operations:

1) composition: \( h(x) = f(g(x)) \);

2) differential recursion: \( h(x,0) = f(x), \partial_y h(x,y) = g(x,y,h(x,y)) \) (the equivalent formulation can be given by integrals: \( h(x,y) = f(x) + \int_0^y g(x,y',h(x,y'))dy' \));

3) \( \mu \)-recursion \( h(x) = \mu_x f(x,y) = \inf \{ y : f(x,y) = 0 \} \), where infimum chooses the number \( y \) with the smallest absolute value and for two \( y \) with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if \( h \) is defined by a differential recursion then \( h \) is defined only where a finite and unique solution exists. This is why the set of \( R \)-recursive functions includes also partial functions. We use (after [4]) the name of \( R \)-recursive functions in the article, however we should remember that in reality we have partiality here (partial \( R \)-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive \( y \) or just below some negative \( y \) then the infimum operation returns that \( y \) even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] \( \mu \)-operation is replaced by infinite limits: \( h(x) = \liminf_{y \to \infty} g(x,y) \), \( h(x) = \limsup_{y \to \infty} g(x,y) \) then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form \( h(x) = \lim_{y \to \infty} g(x,y) \), which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of $R$-recursive functions is closed under the operations of infinite limits: 
$$h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x}, y), \quad h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x}, y), \quad h(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y).$$

3. Hierarchies

The operator $\mu$ is a key operator in generating the $R$-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of $\mu$ in the definition of a given $f$.

Definition 3.1 ([4]) For a given $R$-recursive expression $s(\bar{x})$, let $M_\mu(s)$ (the $\mu$-number with respect to $x_i$) be defined as follows:

$$M_\mu(0) = M_\mu(1) = M_\mu(-1) = 0, \quad (1)$$

$$M_\mu(f(g_1, g_2, \ldots)) = \max_j \left( M_\mu(f) + M_\mu(g_j) \right), \quad (2)$$

$$M_\mu(h = f + \int_0^y g(\bar{x}, y', h) dy') = \max \left( M_\mu(f), M_\mu(g), M_h(g) \right), \quad (3)$$

$$M_\mu(h = f + \int_0^y g(\bar{x}, y', h) dy') = \max \left( M_\mu(f), M_\mu(g), M_h(g) \right), \quad (4)$$

$$M_\mu(\mu, f(\bar{x}, y)) = \max \left( M_\mu(f), M_\mu(f) \right) + 1, \quad (5)$$

where $x$ can be any $x_1, \ldots, x_n$ for $\bar{x} = (x_1, \ldots, x_n)$.

For an $R$-recursive function $f$, let $M(f) = max_\mu(s)$ minimized over all expressions $s$ that define $f$. Now we are ready to define $M$-hierarchy ($\mu$-hierarchy) as a family of $M_j = \{ f : M'(f) \leq j \}$.

Let us construct the analogous definition of $L$-hierarchy by replacing in the above definition $M_\mu$ by $L_\mu$ and changing line (5) to the following form (5'):

$$L_\mu \left( \liminf_{y \to \infty} g(\bar{x}, y) \right) = L_\mu \left( \limsup_{y \to \infty} g(\bar{x}, y) \right) =$$

$$= L_\mu \left( \lim g(\bar{x}, y) \right) = \max \left( L_\mu(f), L_\mu(f) \right) + 1.$$

For an $R$-recursive function $f$, let $L(f) = max_\mu L_\mu(s)$ minimized over all expressions $S$ that define $f$ without using the $\mu$-operation.
Definition 3.2 The $L$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $\mathbb{R}$-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, $xy$, $x / y$, $e^x$, $\ln x$, $y^x$, $\sin x$, $\cos x$ are primitive $\mathbb{R}$-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

Proof. It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi / 2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi / 2, & \text{if } x < 0, \end{cases}$ we obtain $\sgn(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2\arctan 1}$ and $|x| = \sgn(x)x$.

We should be careful with definitions of functions by cases:

Lemma 3.5 For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ \cdots, & \text{if } f(\bar{x}) = k-1, \\ g_k(\bar{x}), & \text{if } f(\bar{x}) \geq k-1 \end{cases}$ and $g_i \in L_n$ for all $1 \leq i \leq k$, $f \in L_m$ the function $h$ belongs to $L_{\max(n_1, \ldots, n_k, m+1)}$. 

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Proof. Let us see that $eq(x, y) = \delta(x - y) \in L_1$ and $ge(x, y) = \frac{\left(\text{sgn}(x - y) + eq(x, y)\right)}{2} + \frac{1}{2} \in L_1$. Then of course

$$h(x) = \sum_{i=1}^{k-1} g_i(x) eq(f(x), i - 1) + g_k(x) ge(f(x), k - 1) \square$$

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 The function $\Theta(x)$ (equal to 1 if $x \geq 0$, otherwise 0), maximum $\max(x, y)$, square-wave function $s$ are in $L_2$, the function $p(x)$ such that $p(x) = 1$ for $x \in [2n, 2n+1]$ and $p(x) = 0$ for $x \in [2n+1, 2n+2]$ is in $L_2$ and the floor function $[x]$ is in $L_3$.

Proof. We give the proper definitions (from [6]) for these functions. Let

$$\Theta(x) = \delta(x - |x|),$$

$$\max(x, y) = x\delta(x - y) + (1 - \delta(x - y))[x\Theta(x - y) + y\Theta(y - x)],$$

$$s(x) = \Theta(\sin(\pi x)).$$

The function $p(x)$ can be given as $s(x)\left(1 - \delta\left(\sin\left(\frac{x-1}{2}\right)\right)\right)$, so $p \in L_2$.

The floor function we can define by the auxiliary function $w(0) = 0$, $\partial_x w(x) = 2\Theta(-\sin(2\pi x))$ as

$$[x] = \begin{cases} 2w(x/2) & \text{if } p(x) = 1, \\ 2w((x-1)/2) & \text{if } p(x) = 0. \end{cases}$$

From the above equation we have $\lfloor x \rfloor$ in $L_3 \square$

Let us recall that if $f : R^n \rightarrow R$ is an $R$-recursive function then the function $f_{iter}(i, \overline{x})$ is $R$-recursive, too.

Lemma 3.7 Let $f : R^n \rightarrow R$ belongs to the class $L_i$, then we have $f_{iter} : R^{n+1} \rightarrow R$ is in $L_{\max(2, j)}$.

Proof. The definitions, which were given by Moore [3] $f_{iter}(i, \overline{x}) = h(2i)$, where

$$h(0) = g(0) = \overline{x},$$
\[ \partial_t g(t) = \left[ f(h(t)) - h(t) \right] s(t), \]
\[ \partial_t h(t) \geq \left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)), \]

with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0 \), \( \partial, r(t) = 2s(t) - 1 \), \( r, s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( \mathbb{R}^l \)-recursive functions \( \gamma_2 : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \gamma_1^1, \gamma_2^2 : \mathbb{R} \rightarrow \mathbb{R} \) such that \((\forall x, y \in \mathbb{R})\gamma_2^1(\gamma_2(x, y)) = x, \quad (\forall x, y \in \mathbb{R})\gamma_2^2(\gamma_2(x, y)) = y, \) have the following properties: \( \gamma_2, \gamma_1^2 \) are in \( L_{10} \), \( \gamma_2^2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2, \Gamma_1^1, \Gamma_2^2 \), which are coding and decoding functions in the interval \((0,1) : \Gamma_2(x, y) = c(x) + c(y)/10 \) , where
\[ c(x) = \lim_{i \to \infty} z \left( a(i, x) \right) \big/ 10^i \] and later
\[ z(x) = \lim_{i \to \infty} z_{iter}^\prime(i, x), \]
\[ z_{iter}^\prime(i, a_1...a_n.a_{n+1}...) = a_1...a_n0...a_{n+1}0.a_{n+1}..., \]
\[ a(i, 0.a_1a_2...a_i...) = 0.a_1...a_i, \]
\[ b(i, 0.a_1a_2...a_i...) = 0.\xi a_{i+1}... \]

\( (z'(x)) = \begin{cases} 100[x] + 10(x - [x]), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x, \end{cases} \]

\( \in L_4, a, b \in L_4 \). Also \( z_{iter}^\prime \) belongs to \( L_4 \), hence \( \Gamma_2(x, y) \in L_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma_1^1(x) \) is in \( L_{10} \), but \( \Gamma_2^2(x) = \Gamma_2^2(10 - \lfloor 10x \rfloor) \) so \( \Gamma_2^2 \in L_{14} \).

The functions \( \Gamma_2, \Gamma_1^1, \Gamma_2^2 \) can be extended to all reals by one-to-one \( f : (0,1) \to R \in L_0 \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( \gamma_1^n : \mathbb{R} \rightarrow \mathbb{R} \) for \( i = 1,...,n \) such that
\[ (\forall i)(\forall x_1,...,x_n \in R)\gamma_1^n(\gamma_n(x_1,...,x_n)) = x_i \]

in the same class: \( \gamma_n, \gamma_1^n \in L_{10} \) and \( (\forall i > 1)\gamma_1^n \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant $p \in N$ that for the function

$$\prod_{z=0}^{y} f(\bar{x}, z) = \begin{cases} f(\bar{x}, 0) f(\bar{x}, 1) \ldots f(\bar{x}, y-1), & \text{if } y \geq 1, \\ 1, & \text{if } 0 \leq y < 1, \\ 0, & \text{if } y < 0, \end{cases}$$

if the function $f$ is in the class $L_m$ then $\prod_{z=0}^{y} f(\bar{x}, z)$ is in the class $L_{m+p}$ ($p$ is independent of $m$).

**Proof.** By the definitions

$$t(w) = \gamma_{n+2} \left( \gamma_{n+2}^{1,n}(w), \gamma_{n+2}^{n+1}(w) + 1, f \left( \gamma_{n+2}^{1,n}(w), \gamma_{n+2}^{n+1}(w) \right) \gamma_{n+2}^{n+1}(w) \right)$$

and

$$S(\bar{x}, z) = t_{\left[ z \right]} \left( s (\bar{x}, 0) \ldots \right) = t_{\left[ z \right]} \left( \left[ z \right], \gamma_{n+2}(\bar{x}, 0,1) \right)$$

we get the property

$$\prod_{y=0}^{z} f(\bar{x}, y) = \gamma_{n+2}^{n+2} \left( S(\bar{x}, z) \right).$$

From the definition of the limit hierarchy we get $\prod_{y=0}^{z} f(\bar{x}, y) \in L_{m+38} \square$.

In the rest of the paper we will use the constant $p$ as the number of limits used in the recursive definition of the product $\prod_{y=0}^{z} f(\bar{x}, y)$ instead of the value 38. The above constructions are tedious and can be improved with a better approximation of $p$.

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between $L$ -hierarchy and $M$ -hierarchy.

**Theorem 4.1** Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $\mathbb{R}$-recursive function. Then if $f \in L_i$ then $f \in M_{10i}$.

**Proof.** We use a simple induction here. The case $i = 0$ is given in Lemma 3.3. Now let us suppose that the thesis is true for $i = n$. Let $f \in L_{n+1}$ be defined as $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ for $g \in L_n$. Then we can recall Theorem 4.2 from [6] which gives us the following result: to define $f$ from $g$ it is necessary to use at
most 10 μ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \).

Similar inferences hold for \( \lim \inf, \lim \sup \).

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(\overline{x},y):R^{n+1} \to R \) is in the class \( L_n \) then the function \( g:R^n \to R, \ g(\overline{x})=\mu_y f(\overline{x},y) \) is in the class \( L_{n+3,9} \) as from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function \( g:R^n \to R, \ g(\overline{x})=\mu_y f(\overline{x},y) \) for \( f(\overline{x},y):R^{n+1} \to R \) (\( f \) - \( R \)-recursive) replacing the \( \mu \)-operator by limit operation. First we introduced the function

\[
Z' (\overline{x},z) = \begin{cases} 
\inf_y \left\{ f : K' (\overline{x}, y) = 0 \right\}, & \text{if } z = 0 \text{ and } \exists y K' (\overline{x}, y) = 0, \\
\text{undefined} & \text{if } z = 0 \text{ and } \forall y K' (\overline{x}, y) \neq 0, \\
1 & \text{if } z \neq 0,
\end{cases}
\]

given in the following way:

\[
Z' (\overline{x},z) = \begin{cases} 
\text{undefined} & \text{if } (z=0) \land \left( S' (\overline{x}) < \frac{1}{12} \right), \\
\sqrt{S' (\overline{x})-\frac{1}{12}} & \text{if } (z=0) \land \left( S' (\overline{x}) \geq \frac{1}{12} \right) \\
\wedge f (\overline{x},\sqrt{S' (\overline{x})-\frac{1}{12}}) = 0, & \text{if } (z=0) \land \left( S' (\overline{x}) \geq \frac{1}{12} \right) \\
-\sqrt{S' (\overline{x})-\frac{1}{12}} & \text{if } (z=0) \land \left( S' (\overline{x}) \geq \frac{1}{12} \right) \\
\wedge f (\overline{x},-\sqrt{S' (\overline{x})-\frac{1}{12}}) = 0, & \text{if } z \neq 0.
\end{cases}
\]

where \( S' (\overline{x}) = \lim_{t \to -\infty} S'_i (\overline{x},t) + \lim_{t \to +\infty} S'_i (\overline{x},t) \). Both functions \( S'_1 \), \( S'_2 \) are defined by an integration

\[
S'_i (\overline{x},t) = \int y^2 \left( 1 - h'^i (\overline{x},(-1)^{i+1} y - 1/2,(-1)^{i+1} y + 1/2) \right) dy, \ i=1,2
\]

from \( h'^i (\overline{x},a,b) = \liminf_{t \to \infty} \prod_{w=0}^{i+1} K' \left( \overline{x},a + w \frac{b-a}{z} \right) \) where \( K' \) is the characteristic function of \( f \).

Hence we can conclude that if \( K' \) is in the \( L_s \) then \( Z_f \) is in the class \( L_{s+p+3} \).

Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5].
\[ K^f_\mu(y) = 1 - \lim_{x \to -\infty} \lim_{z \to +\infty} G^f(x, z, a, b, y), \]

where \( G^f(x, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{[z]}\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a, b]\) and value 0 otherwise. Precisely for \(y\) from \([a, a + \frac{b-a}{2^{[z]}}]\)

\[
G^f(x, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f(x, a + \frac{(k-1)(b-a)}{2^{|z|}}, a + \frac{k(b-a)}{2^{|z|}}) = 0, \\
0, & \text{otherwise} 
\end{cases}
\]

for \(y \in \left(a + \frac{(k-1)(b-a)}{2^{|z|}}, a + \frac{k(b-a)}{2^{|z|}}\right)\) (where \(k = 2, 3, ..., 2^n\)) we have:

\[
G^f(x, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f(x, a + \frac{(i-1)(b-a)}{2^{|z|}}, a + \frac{i(b-a)}{2^{|z|}}) \neq 0, \\
\land h^f(x, a + \frac{(k-1)(b-a)}{2^{|z|}}, a + \frac{k(b-a)}{2^{|z|}}) = 0, & \text{otherwise} 
\end{cases}
\]

and for \(Y \notin [A, B]\) the function \(g^f_x\) is equal to 2.

The definition of \(G_f\) is given by the cases with respect to the value of the expression given by \(\prod h^f\), since for \(f \in L_m\), the function \(h_f \in L_{m+p+2}\) and \(G^f \in L_{m+2,p+3}\). Then we have \(K^f_\mu \in L_{m+2,p+6}\). Now we must use the function \(K^f_\mu\) in the same way as \(K^f\) which gives us \(Z_f\) in the class \(L_{m+3,p+9}\). The final definition of \(g(x) = \mu, f(x, y)\) ([5] Theorem 4.3) given below...
\[
g(x) = \begin{cases} 
Z^f(x,0) - Z^f(x,0), & \text{if } S^f(x) < \frac{1}{12} \land S^f(x) < \frac{1}{12}, \\
Z^f(x,0), & \text{if } \left(S^f(x) \geq \frac{1}{12} \land S^f(x) < \frac{1}{12}\right) \\
- Z^f(x,0), & \text{if } \left(S^f(x) < \frac{1}{12} \land S^f(x) \geq \frac{1}{12}\right) \\
\land \left(Z^f(x,0) < Z^f(x,0)\right), & \text{or} \\
\land \left(Z^f(x,0) \geq Z^f(x,0)\right), & \text{or} \\
\end{cases}
\]

where 
\[
f^+(x, y) = \begin{cases} 
f(x, y), & y \geq 0, \\
1, & y < 0; 
\end{cases}
\]
\[
f^-(x, y) = \begin{cases} 
f(x, -y), & y > 0, \\
1, & y \leq 0; 
\end{cases}
\]
remains the class of g identical to the class of \( Z^f \), i.e. \( g \in L_{m+3,p+9} \).

**Theorem 4.3** Let \( f : R^n \to R \) be an \( R \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

**5. Conclusions**

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum^0_n$-measurable functions and $\mathbb{R}$-recursive functions is an open problem.

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**References**