Two hierarchies of R-recursive functions

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Abstract

In the paper some aspects of complexity of R-recursive functions are considered. The limit hierarchy of R-recursive functions is introduced by the analogy to the \( \mu \)-hierarchy. Then its properties and relations to the \( \mu \)-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals \( \mathbb{R} \) (called R-recursive functions) in the analogous way to the classical recursive functions on the natural numbers \( \mathbb{N} \). His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation \( \mu \), which is used to construct \( \mu \)-hierarchy of R-recursive functions.

It was shown [5] that the zero-finding operator \( \mu \) can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to \( \mu \)-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called R-recursive functions [4].

**Definition 2.1** The set of R-recursive functions is generated from the constants 0,1 by the operations:

1) composition: \( h(\bar{x}) = f\left(g(\bar{x})\right) \);

2) differential recursion: \( h(\bar{x},0) = f(\bar{x}), \partial, h(\bar{x},y) = g(\bar{x},y,h(\bar{x},y)) \) (the equivalent formulation can be given by integrals: \( h(\bar{x},y) = f(\bar{x}) + \int_0^\gamma g(\bar{x},y',h(\bar{x},y'))dy' \));

3) \( \mu \)-recursion \( h(\bar{x}) = \mu, f(\bar{x},y) = \inf \{y : f(\bar{x},y) = 0\} \), where infimum chooses the number \( y \) with the smallest absolute value and for two \( y \) with the same absolute value the negative one;

4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if \( h \) is defined by a differential recursion then \( h \) is defined only where a finite and unique solution exists. This is why the set of R-recursive functions includes also partial functions. We use (after [4]) the name of R-recursive functions in the article, however we should remember that in reality we have partiality here (partial R-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive \( y \) or just below some negative \( y \) then the infimum operation returns that \( y \) even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] \( \mu \)-operation is replaced by infinite limits: \( h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x},y) \), \( h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x},y) \) then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form \( h(\bar{x}) = \lim_{y \to \infty} g(\bar{x},y) \), which can be in the obvious way obtained from limsup, liminf:
Corollary 2.2 The class of \( R \)-recursive functions is closed under the operations of infinite limits: 
\[
  h(\bar{x}) = \liminf_{y \to \infty} g(\bar{x}, y), \quad h(\bar{x}) = \limsup_{y \to \infty} g(\bar{x}, y), \\
  h(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y).
\]

3. Hierarchies

The operator \( \mu \) is a key operator in generating the \( R \)-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of \( \mu \) in the definition of a given \( f \).

Definition 3.1 ([4]) For a given \( R \)-recursive expression \( s(\bar{x}) \), let \( M_x(s) \) (the \( \mu \)-number with respect to \( x_j \)) be defined as follows:

\[
  M_x(0) = M_x(1) = M_x(-1) = 0, \\
  M_x(f(g_1, g_2, \ldots)) = \max \left( M_{x_j}(f) + M_{x_j}(g_j) \right), \\
  M_x(h = f + \int_0^\infty g(\bar{x}, y', h)dy') = \max \left( M_x(f), M_x(g), M_h(g) \right), \\
  M_y(h = f + \int_0^\infty g(\bar{x}, y', h)dy') = \max \left( M_y(g), M_h(g) \right), \\
  M_x(\mu, f(\bar{x}, y)) = \max \left( M_x(f), M_y(f) \right) + 1,
\]

where \( x \) can be any \( x_1, \ldots, x_n \) for \( \bar{x} = (x_1, \ldots, x_n) \).

For an \( R \)-recursive function \( f \), let \( M(f) = \max_x(M_x(s)) \) minimized over all expressions \( s \) that define \( f \). Now we are ready to define \( M \)-hierarchy (\( \mu \)-hierarchy) as a family of \( M_j = \{ f : M_j(f) \leq j \} \).

Let us construct the analogous definition of \( L \)-hierarchy by replacing in the above definition \( M_x \) by \( L_x \) and changing line (5) to the following form (5'):

\[
  L_x(\liminf_{y \to \infty} g(\bar{x}, y)) = L_x(\limsup_{y \to \infty} g(\bar{x}, y)) = L_x(\lim_{y \to \infty} g(\bar{x}, y)) = \max \left( L_x(f), L_y(f) \right) + 1.
\]

For an \( R \)-recursive function \( f \), let \( L(f) = \max_x(L_x(s)) \) minimized over all expressions \( S \) that define \( f \) without using the \( \mu \)-operation.
**Definition 3.2** The $L$-hierarchy is a family of $L_j = \{ f : L(f) \leq j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would effect in a superficially higher class of a complexity of a function $f$.

As an obvious corollary from definitions we have the following statement.

**Lemma 3.3** The classes $M_0$ and $M_1$ are identical.

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive $R$-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x + y$, $xy$, $x/y$, $e^x$, $\ln x$, $y^x$, $\sin x$, $\cos x$ are primitive $R$-recursive.

We can give a few results on some levels of the limit hierarchy.

**Lemma 3.4.** The Kronecker $\delta$ function, the signum function and absolute value belong to the first level ($L_1$) of limit hierarchy.

**Proof.** It is sufficient to take the following definitions [5]: hence $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$ let us define $\delta(x) = \liminf_{y \to \infty} \left( \frac{1}{1 + x^2} \right)^y$. Now from the expression $\liminf_{y \to \infty} \arctan xy = \begin{cases} \pi/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\pi/2, & \text{if } x < 0, \end{cases}$ we obtain $\sgn(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2\arctan 1}$ and $|x| = \sgn(x)x$.

We should be careful with definitions of functions by cases:

**Lemma 3.5** For $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \text{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \text{if } f(\bar{x}) = 1, \\ M & \text{if } f(\bar{x}) \geq k - 1 \end{cases}$ and $g_i \in L_i$ for all $1 \leq i \leq k$, $f \in L_m$ the function $h$ belongs to $L_{\max(n_1,...,n_k,m+1)}$. 


Proof. Let us see that \( \text{eq}(x, y) = 1 \) for \( x \geq 0 \), otherwise 0, maximum \( \max(x, y) \), square-wave function \( s \) are in \( L_2 \), the function \( p(x) \) such that \( p(x) = 1 \) for \( x \in [2n, 2n + 1] \) and \( p(x) = 0 \) for \( x \in [2n + 1, 2n + 2] \) is in \( L_2 \) and the floor function \( \lfloor x \rfloor \) is in \( L_3 \).

Proof. We give the proper definitions (from [6]) for these functions. Let
\[
\Theta(x) = \delta(x - |x|),
\]
\[
\max(x, y) = x\delta(x - y) + (1 - \delta(x - y))\left[x\Theta(x - y) + y\Theta(y - x)\right],
\]
\[
s(x) = \Theta(\sin(\pi x)).
\]

The function \( p(x) \) can be given as
\[
s(x)\left(1 - \delta\left(\frac{\sin(\frac{x - 1}{2}\pi)}{2}\right)\right),
\]
so \( p \in L_2 \).

The floor function we can define by the auxiliary function \( w(0) = 0, \partial_x w(x) = 2\Theta(-\sin(2\pi x)) \) as
\[
\lfloor x \rfloor = \begin{cases} 
2w(x/2) & \text{if } p(x) = 1, \\
2w((x - 1)/2) & \text{if } p(x) = 0.
\end{cases}
\]

From the above equation we have \( \lfloor x \rfloor \) in \( L_3 \). □

Let us recall that if \( f : R^n \rightarrow R \) is an \( R \)-recursive function then the function \( f_{\text{iter}}(i, \bar{x}) \) is \( R \)-recursive, too.

Lemma 3.7 Let \( f : R^n \rightarrow R \) belongs to the class \( L_i \), then we have \( f_{\text{iter}} : R^{n+1} \rightarrow R \) is in \( L_{\text{max}}(2, j) \).

Proof. The definitions, which were given by Moore [3] \( f_{\text{iter}}(i, \bar{x}) = h(2i) \), where
\[
h(0) = g(0) = \bar{x},
\]
\[ \partial_t g(t) = \left[ f(h(t)) - h(t) \right] s(t), \]
\[ \partial_t h(t) = -\left[ \frac{g(t) - h(t)}{r(t)} \right] (1 - s(t)), \]

with \( s \) - a square wave function in \( L_2 \) and \( r(0) = 0 \), \( \partial_t r(t) = 2s(t) - 1 \), \( r,s \in L_2 \) give us the desirable statement. □

**Lemma 3.8** The \( R^l \)-recursive functions \( \gamma_2 : R^2 \to R \), \( \gamma_1^2, \gamma_2^2 : R \to R \) such that \( (\forall x, y \in R) \gamma_2^1 \left( \gamma_2^2 (x, y) \right) = x \), \( (\forall x, y \in R) \gamma_2^2 \left( \gamma_2^1 (x, y) \right) = y \), have the following properties: \( \gamma_2 \), \( \gamma_1^2 \) are in \( L_{10} \), \( \gamma_2^2 \) is in \( L_{14} \).

**Proof.** We have the auxiliary functions \( \Gamma_2 \), \( \Gamma_1^2 \), \( \Gamma_2^2 \), which are coding and decoding functions in the interval \((0,1): \Gamma_2^2 (x, y) = c(x) + c(y)/10 \), where
\[ c(x) = \lim_{i \to \infty} z(i \cdot x), \]
and later \( z(x) = \lim_{i \to \infty} z_{\text{iter}}^i (i \cdot x), \)
\[ z_{\text{iter}}^i (i \cdot a_1, a_2, \ldots, a_n) = a_1 \ldots a_n 0 \ldots a_{n+1} \ldots, \]
\[ a(i, 0, a_1, a_2, \ldots) = 0.a_1 \ldots a_i \]
\[ b(i, 0, a_1, a_2, \ldots) = 0.0a_1 \ldots a_i \ldots, \]

\[ (z'(x)) = \begin{cases} 100 \lfloor x \rfloor + 10 \left( x - \lfloor x \rfloor \right), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x \end{cases} \]
\[ \in L_4, a,b \in L_4. \]

Also \( z_{\text{iter}} \) belongs to \( L_4 \), hence \( \Gamma_2(x, y) \in L_{10} \), decoding of the first element is described in the symmetric way so \( \Gamma_2^1(x) \) is in \( L_{10} \), but \( \Gamma_2^2(x) = \Gamma_2^2 (10 - \lfloor 10x \rfloor) \) so \( \Gamma_2^2 \in L_{14} \).

The functions \( \Gamma_2 \), \( \Gamma_1^2 \), \( \Gamma_2^2 \) can be extended to all reals by one-to-one \( f : (0,1) \to R \in L_6 \) without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions \( \gamma_n : R^n \to R \) and \( \gamma_n^i : R \to R \) for \( i = 1, \ldots, n \) such that
\[ (\forall i) (\forall x_1, \ldots, x_n \in R) \gamma_n^i \left( \gamma_n \left(x_1, \ldots, x_n \right) \right) = x_i \]
in the same class: \( \gamma_n, \gamma_n^i \in L_{10} \) and \( (\forall i > 1) \gamma_n^i \in L_{14} \).

We finish this part with the important form of defining: a new function is given as a product of values \( f \) in some integer points.
Lemma 3.9 There exists such constant $p \in \mathbb{N}$ that for the function
\[
\prod_{z=0}^{y} f(\bar{x}, z) = \begin{cases} 
  f(\bar{x}, 0) f(\bar{x}, 1) \ldots f(\bar{x}, \lfloor y - 1 \rfloor), & \text{if } y \geq 1, \\
  1, & \text{if } 0 \leq y < 1, \\
  0, & \text{if } y < 0,
\end{cases}
\]
if the function $f$ is in the class $L_m$ then $\prod_{z=0}^{y} f(\bar{x}, z)$ is in the class $L_{m+p}$ ($p$ is independent of $m$).

Proof. By the definitions
\[
t(w) = \gamma_n \left( \gamma_n(1, w), \gamma_{n+1}(2, w) + 1, f \left( \gamma_n(1, w), \gamma_{n+1}(2, w) \right) \cdot \gamma_{n+2}(3, w) \right)
\]
and
\[
S(\bar{x}, z) = t_{\lfloor z \rfloor} \left( \lim_{y \to \infty} f(\bar{x}, y) \right) = t_{\lfloor z \rfloor} \left( \gamma_n(1, \bar{x}, 0, 1) \right)
\]
we get the property
\[
\prod_{y=0}^{z} f(\bar{x}, y) = \gamma_{n+2} \left( S(\bar{x}, z) \right).
\]
From the definition of the limit hierarchy we get $\prod_{y=0}^{z} f(\bar{x}, y) \in L_{m+38}$.

In the rest of the paper we will use the constant $p$ as the number of limits used in the recursive definition of the product $\prod_{y=0}^{z} f(\bar{x}, y)$ instead of the value 38. The above constructions are tedious and can be improved with a better approximation of $p$.

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between $L$-hierarchy and $M$-hierarchy.

Theorem 4.1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be an R-recursive function. Then if $f \in L_i$ then $f \in M_{10i}$.

Proof. We use a simple induction here. The case $i = 0$ is given in Lemma 3.3. Now let us suppose that the thesis is true for $i = n$. Let $f \in L_{n+1}$ be defined as $f(\bar{x}) = \lim_{y \to \infty} g(\bar{x}, y)$ for $g \in L_n$. Then we can recall Theorem 4.2 from [6] which gives us the following result: to define $f$ from $g$ it is necessary to use at
most 10 μ-operation. Hence for \( g \in M_{10n} \) the function \( f \) satisfies \( f \in M_{10n+10} \). Similar inferences hold for \( \lim \inf \), \( \lim \sup \).

Now we can give the result about the 'limit complexity' of the infimum operator \( \mu \).

**Lemma 4.2** If \( f(x,y): R^{n+1} \to R \) is in the class \( L_m \) then the function \( g: R^n \to R, \quad g(x) = \mu_y f(x,y) \) is in the class \( L_{m+3p+9} \) is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function \( g: R^n \to R, \quad g(x) = \mu_y f(x,y) \) for \( f(x,y): R^{n+1} \to R \) (\( f \) - \( R \)-recursive) replacing the \( \mu \)-operator by limit operation. First we introduced the function

\[
Z^f(x,z) = \begin{cases} 
\inf_y \{ f : K^f(x,y) = 0 \}, & \text{if } z = 0 \text{ and } \exists y K^f(x,y) = 0, \\
\text{undefined} & \text{if } z = 0 \text{ and } \forall y K^f(x,y) \neq 0, \\
1 & \text{if } z \neq 0,
\end{cases}
\]

given in the following way:

\[
Z^f(x,z) = \begin{cases} 
\text{undefined} & \text{if } (z=0) \land \left( S^f(x) < \frac{1}{12} \right), \\
\sqrt{S^f(x)} - \frac{1}{12}, & \text{if } (z=0) \land \left( S^f(x) \geq \frac{1}{12} \right) \land f(x,\sqrt{S^f(x)} - \frac{1}{12}) = 0, \\
-\sqrt{S^f(x)} - \frac{1}{12}, & \text{if } (z=0) \land \left( S^f(x) \geq \frac{1}{12} \right) \land f(x,-\sqrt{S^f(x)} - \frac{1}{12}) = 0, \\
1, & \text{if } z \neq 0,
\end{cases}
\]

where \( S^f(x) = \lim_{t\to\infty} S^f_1(x,t) + \lim_{t\to\infty} S^f_2(x,t) \). Both functions \( S^f_1, S^f_2 \) are defined by an integration

\[
S^f_1(x,t) = \int y^2 \left( 1 - h^f(x,-1)^{i+1} y - 1/2, (-1)^{i+1} y + 1/2 \right) dy, \quad i = 1,2
\]

from \( h^f(x,a,b) = \liminf_{y\to\infty} \prod_{w=0}^{x+1} K^f \left( x, a + w \frac{b-a}{z} \right) \) where \( K^f \) is the characteristic function of \( f \).

Hence we can conclude that if \( K^f \) is in the \( L_s \) then \( Z^f \) is in the class \( L_{s+p+3} \). Let us finish with the definition of the characteristic function of the infimum of zeros of \( f \) (see Theorem 4.2 from [5]
\[ K^f_\mu(y) = 1 - \lim_{a \to -\infty} \lim_{b \to \infty} \lim_{z \to \infty} G^f(x, z, a, b, y), \]

where \( G^f(x, z, a, b, y) \) divides the interval \([a, b]\) into \(2^{[z]}\) equal subintervals and gives the value 1 for \(y\) from the subintervals, which contains the least zero of \(f\) in \([a, b]\) and value 0 otherwise. Precisely for \(y\) from \([a, a + \frac{b-a}{2^{[z]}}]\)

\[
G^f(x, z, a, b, y) = \begin{cases} 
1, & \text{if } h^f(x, a, a + \frac{b-a}{2^{[z]}}) = 0, \\
0, & \text{otherwise}
\end{cases}
\]

for \(y \in \left(a + \frac{(k-1)(b-a)}{2^{[z]}}, a + \frac{k(b-a)}{2^{[z]}}\right)\) (where \(k = 2, 3, \ldots, 2^n\)) we have:

\[
G^f(x, z, a, b, y) = \begin{cases} 
1, & \text{if } \prod_{i=1}^{k-1} h^f(x, a + \frac{(i-1)(b-a)}{2^{[z]}}, a + \frac{i(b-a)}{2^{[z]}}) \neq 0, \\
\wedge h^f(x, a + \frac{(k-1)(b-a)}{2^{[z]}}, a + \frac{k(b-a)}{2^{[z]}}) = 0, & \text{otherwise}
\end{cases}
\]

and for \(y \notin [A, B]\) the function \(g^f_x\) is equal to 2.

The definition of \(G_f\) is given by the cases with respect to the value of the expression given by \(\prod h^f\), since for \(f \in L_m\), the function \(h_f \in L_{m+p+2}\) and \(G^f \in L_{m+2,p+3}\). Then we have \(K^f_\mu \in L_{m+2,p+6}\). Now we must use the function \(K^f_\mu\) in the same way as \(K^f\) which gives us \(Z_f\) in the class \(L_{m+3,p+9}\). The final definition of \(g(x) = \mu_x f(x, y)\) ([5] Theorem 4.3) given below.
\[
g(\bar{x}) = \begin{cases} 
Z^{f^+}(\bar{x},0) - Z^{f^-}(\bar{x},0), & \text{if } S^{f^+}(\bar{x}) < \frac{1}{12} \land S^{f^-}(\bar{x}) < \frac{1}{12}, \\
Z^{f^+}(\bar{x},0), & \text{if } \left(S^{f^+}(\bar{x}) \geq \frac{1}{12} \land S^{f^-}(\bar{x}) < \frac{1}{12}\right) \\
\lnot Z^{f^-}(\bar{x},0), & \text{if } \left(S^{f^+}(\bar{x}) < \frac{1}{12} \land S^{f^-}(\bar{x}) \geq \frac{1}{12}\right) \\
\lnot Z^{f^-}(\bar{x},0) \land Z^{f^+}(\bar{x},0) < Z^{f^-}(\bar{x},0), & \text{or} \\
\left(S^{f^+}(\bar{x}) < \frac{1}{12} \land S^{f^-}(\bar{x}) < \frac{1}{12}\right) \\
\land Z^{f^+}(\bar{x},0) \geq Z^{f^-}(\bar{x},0)), & \text{or} \\
\left(S^{f^+}(\bar{x}) \geq \frac{1}{12} \land S^{f^-}(\bar{x}) < \frac{1}{12}\right) \\
\land Z^{f^+}(\bar{x},0) \lnot Z^{f^-}(\bar{x},0)), & \text{or} \\
\end{cases}
\]

where \( f^+(\bar{x}, y) = \begin{cases} 
f(\bar{x}, y), & y \geq 0, \\
1, & y < 0;
\end{cases}\)

\( f^-(\bar{x}, y) = \begin{cases} 
f(\bar{x}, -y), & y > 0, \\
1, & y \leq 0;
\end{cases}\)

remains the class of \( g \) identical to the class of \( Z^f \), i.e. \( g \in L_{m+3,p+9} \).

\textbf{Theorem 4.3} Let \( f : R^n \to R \) be an \( R \)-recursive function. Then for all \( i \geq 0 \) if \( f \in M_i \) then \( f \in L_{(i,p+9)i} \).

The above statement is a simple consequence of the fact \( M_0 = L_0 \) and Lemma 4.2.

\textbf{5. Conclusions}

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the \( \mu \)-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a \( \mu \)-operator.

We also establish the proper relation between the levels of the limit hierarchy and \( \mu \)-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also \( \mu \)-hierarchy) and Baire classes.
[7]. Also the kind of a connection between the $\sum^0_n$-measurable functions and $\mathbb{R}$-recursive functions is an open problem.

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References