Newton-like method for singular 2-regular system of nonlinear equations

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Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

1. Introduction

Let $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^* \in D$ of the equation

$$F(x) = 0.$$  

(1)

Definition 1

A linear operator $\Psi_2(h) : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $h \in \mathbb{R}^n$ is called 2-factor operator, if

$$\Psi_2(h) = F'(x^*) + P^\perp F'(x^*) h,$$

(2)

where

$P^\perp$ denotes the orthogonal projection on $\left(\text{Im } F'(x)\right)^\perp$ in $\mathbb{R}^n$ [1].

Definition 2

Operator $F$ is called 2-regular in $x^*$ on the element $h \in \mathbb{R}^n$, $h \neq 0$, if the operator $\Psi_2(h)$ has the property:

$$\text{Im } \Psi_2(h) = \mathbb{R}^m.$$
Definition 3
Operator F is called 2-regular in $x^*$, if F is 2-regular on the set $K_2(x^*)\{0\}$, where

$$K_2(x^*) = \text{Ker}F^-(x^*) \cap \text{Ker}^2 P^+ F^-(x^*),$$

(3)

$$\text{Ker}^2 P^+ F^-(x^*) = \{h \in R^n : P^+ F^-(x^*)[h]^2 = 0\}.$$

We need the following assumption on F:

A1) completely degenerated in $x^*$:

$$\text{Im} F^-(x^*) = 0.$$  
(4)

A2) operator F is 2-regular in $x^*$:

$$\text{Im} F^-(x^*) h = R^m \text{ for } h \in K_2(x^*), h \neq 0.$$  
(5)

A3) 

$$\text{Ker} F^-(x^*) \neq \{0\}.$$  
(6)

If F satisfies A1 in $x^*$, then

$$K_2(x^*) = \text{Ker}^2 F^-(x^*) = \{h \in R^n : F^-(x^*)[h]^2 = 0\}.$$  
(7)

In [1] it was proved, that if $n=m$, then the sequence

$$x_{k+1} = x_k - \left\{ \hat{F}^-(x_k) + P^+_k F^-(x_k) h_k \right\}^{-1} \cdot \left\{ F(x_k) + P^+_k F^-(x_k) h_k \right\},$$  
(8)

where

$P^+_k$ - denotes orthogonal projection on $\left( \text{Im} \hat{F}^-(x_k) \right)^\perp \text{ in } R^n$,

$h_k \in \text{Ker} \hat{F}^-(x_k), \|h_k\| = 1$

converges Q-quadratically to $x^*$.

The matrices $\hat{F}^-(x_k)$ obtained from $F^-(x_k)$ by replacing all elements, whose absolute values do not increase $\nu > 0$, by zero, where $\nu = \nu_k = \| F(x_k) \|^{(1-\alpha)/2}$, $0 < \alpha < 1$.

In the case $n = m+1$ the operator

$$\left\{ \hat{F}^-(x_k) + P^+_k F^-(x_k) h_k \right\}^{-1}$$
in method (8) is replaced by the operator

$$\left[ \hat{F}^-(x_k) + P^+_k F^-(x_k) h_k \right]^\perp$$
(9)

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined ($n > m$) and degenerated in $x^*$. 

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2. Extending of the system of equation

Now we construct the operator $\Phi : R^n \rightarrow R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*) = 0$ [2].

Assume

A4) Let $F(x) = [f_1(x), f_2(x), \ldots, f_m(x)]^T$, $n > m$ is two continuously differentiable in some neighbourhood $U \subset R^n$ of the point $x^*$.

Denote:

$$H = \text{lin}\{h\} \quad \text{for} \quad h \in \text{Ker}^2 F'(x^*), \; h \neq 0.$$

$P = P_H^\perp$ denotes the orthogonal projection $R^n$ on $H^\perp$.

$$f_i^\prime q(x) = P(f_i^\prime (x))^T \quad \text{for} \; i = 1, 2, \ldots, m.$$

For each system of indices $i_1, i_2, \ldots, i_{n-m-1} \subset \{1, 2, \ldots, m\}$ and vectors $h_1, h_2, \ldots, h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x) h \\ \varphi(x) \end{bmatrix},$$

where

$$\varphi(x) : R^n \rightarrow R^r, \quad r = n - m - 1,$$

$$\varphi(x) = PF'F(x)H, \quad H = [h_1, h_2, \ldots, h_r]^T,$$

$$\varphi(x) = M = \begin{bmatrix} f_{i_1}^\prime q(x) h_1 \\ f_{i_2}^\prime q(x) h_1 \\ \vdots \\ f_{i_r}^\prime q(x) h_1 \end{bmatrix}.$$  \hspace{1cm} (11)

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[\Phi'(x_k)\right]^+ \cdot \Phi(x_k), \quad k = 0, 1, 2, \ldots, (12)$$

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \left(B_k\right)^+ \cdot \Phi(x_k).$$ \hspace{1cm} (13)

The operator $\Phi'$ will be approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k.$$ \hspace{1cm} (14)

We propose matrices $B_k$ which satisfy the secant equation:

$$B_{k+1}s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for} \; k = 0, 1, 2, \ldots, (15)$$

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:
\[ B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \quad \text{for } k=0,1,2,\ldots \] (16)

where

\[ r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \] (17)

We will prove for this method:

**Q-linear convergence** to \( x^* \), i.e. there exists \( q \in (0,1) \) such that

\[ \| x_{k+1} - x^* \| \leq q \| x_k - x^* \| \quad \text{for } k = 0,1,2,\ldots \] (18)

and next Q-superlinear convergence to \( x^* \), i.e.:

\[ \lim_{k \to \infty} \frac{\| x_{k+1} - x^* \|}{\| x_k - x^* \|} = 0. \] (19)

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator \( \Phi' \left( x^* \right) \) is nonsingular.

**Theorem 1** (The Bounded Deterioration Theorem)

Let \( F \) satisfies the assumptions A1-A4. If exist constants \( q_1 \geq 0 \) and \( q_2 \geq 0 \) such that matrices \( \{ B_k \} \) satisfy the inequality:

\[ \| B_{k+1} - \Phi' \left( x^* \right) \| \leq (1 + q_1 r_k) \| B_k - \Phi' \left( x^* \right) \| + q_2 r_k, \] (20)

then there are constants \( \varepsilon > 0 \) and \( \delta > 0 \) such that if

\[ \| x_0 - x^* \| \leq \varepsilon \quad \text{and} \quad \| B_0 - \Phi' \left( x^* \right) \| \leq \delta, \]

then the sequence

\[ x_{k+1} = x_k - B_k^* \Phi \left( x_k \right) \]

converges Q-linearly to \( x^* \).

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

**Theorem 2** (Linear convergence)

Let \( F \) satisfies the assumptions A1-A4. Then the method

\[ x_{k+1} = x_k - \{ B_k \}^* \cdot \Phi \left( x_k \right), \]

\[ B_{k+1} = B_k - \frac{\{ \Phi \left( x_{k+1} \right) - \Phi \left( x_k \right) - B_k s_k \} s_k^T}{s_k^T s_k} \]

locally and Q-linearly converges to \( x^* \).

**Proof.**

To prove the Theorem we should prove the inequality (20) from Theorem 1.

Now we notice:
$$\left\| B_{k+1} - \Phi' \left( x^* \right) \right\| = \left\| B_k - \frac{\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \} s_k^T}{s_k^T s_k} - \Phi' \left( x^* \right) \right\| \leq$$

$$\leq \left\| B_k - \Phi' \left( x^* \right) \right\| + \left\| \left\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \right\} s_k^T \right\| \leq \left\| B_k - \Phi' \left( x^* \right) \right\| +$$

$$+ \left\| \left\{ \Phi(x_{k+1}) - \Phi(x_k) - \Phi' \left( x^* \right) s_k + \Phi \left( x^* \right) s_k - B_k s_k \right\} s_k^T \right\| \leq \left\| B_k - \Phi' \left( x^* \right) \right\| +$$

$$+ \left\| \left( \Phi(x_{k+1}) - \Phi' \left( x^* \right) \right)(x_{k+1} - x^*) s_k^T \right\| + \left\| \left( \Phi(x_k) - \Phi' \left( x^* \right) \right)(x_k - x^*) s_k^T \right\| +$$

$$+ \left\| \left( \Phi' \left( x^* \right) - B_k \right) s_k s_k^T \right\| \leq \left\| \Phi' \left( x^* \right) - B_k \right\| \left( 1 + c_1 r_k \right) + c_1 \left\| x_{k+1} - x^* \right\| \left\| s_k \right\| +$$

$$+ c_2 \left\| x_k - x^* \right\| \left\| s_k \right\| \leq \left\| \Phi' \left( x^* \right) - B_k \right\| \left( 1 + q_1 r_k \right) + q_2 r_k ,$$

where $c_1 > 0$, $c_2 > 0$, $q_1 > 0$, $q_2 > 0$, $r_k = \max \{ \left\| x_{k+1} - x^* \right\|, \left\| x_k - x^* \right\| \}$. □

**Theorem 3 (Q-superlinear convergence)**

Let $F$ satisfies the assumptions A1-A4 and the sequence

$$x_{k+1} = x_k - \{ B_k \}^{-1} \cdot \Phi(x_k),$$

$$B_{k+1} = B_k - \frac{\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \} s_k^T}{s_k^T s_k}$$

linearly converges to $x^*$. Then the sequence $\{ x_k \}$ Q-superlinearly converges to $x^*$.

**Proof.**

Matrices $B_k$ satisfy secant equation (15), so

$$B_{k+1} = P_{L_k} B_k$$

where

$$L_k = \left\{ X : X s_k = y_k, \text{ where } y_k = \Phi' \left( x_{k+1} \right) - \Phi' \left( x_k \right) \right\}$$

Denote

$$H_k = H \left( x_k, x_{k+1} \right) = \int_0^1 \Phi' \left( x_k + t \left( x_{k+1} - x_k \right) \right) dt .$$

We have $H_k \in L_k$ [4].
From (21) and [3] it follows:
\[
\|B_{k+1} - B_k\|^2 + \|B_{k+1} - H_k\|^2 = \|B_k - H_k\|^2, \quad \text{for } i = 0, 1, 2, \ldots .
\]
By lemma 2 [5] we get \(\sum_{k=1}^{\infty} \|B_{k+1} - B_k\|^2 < \infty\), thus we obtain
\[
\|B_{k+1} - B_k\| \rightarrow 0.
\]
This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. \(\square\)

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of \(F^{\prime\prime}(x_k)\).

References