Newton-like method for singular 2-regular system of nonlinear equations

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Abstract
In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear converegence for the nonlinear operators is developed.

1. Introduction
Let $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^* \in D$ of the equation

$$F(x) = 0.$$ (1)

Definition 1
A linear operator $\Psi_2 (h) : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $h \in \mathbb{R}^n$ is called 2-factor operator, if

$$\Psi_2 (h) = F'(x^*) + P^\perp F'(x^*) h,$$ (2)

where $P^\perp$ - denotes the orthogonal projection on $(\text{Im} F'(x))^\perp$ in $\mathbb{R}^n$ [1].

Definition 2
Operator $F$ is called 2-regular in $x^*$ on the element $h \in \mathbb{R}^n$, $h \neq 0$, if the operator $\Psi_2 (h)$ has the property:

$$\text{Im} \Psi_2 (h) = \mathbb{R}^m.$$
Definition 3
Operator F is called 2-regular in $x^*$, if F is 2-regular on the set $K_2(x^*)\setminus\{0\}$, where

$$K_2(x^*) = \text{Ker} F' (x^*) \cap \text{Ker}^2 P^\perp F' (x^*), \quad (3)$$

$$\text{Ker}^2 P^\perp F' (x^*) = \{ h \in \mathbb{R}^n : P^\perp F' (x^*)[h]^2 = 0 \}.$$  

We need the following assumption on F:
A1) completely degenerated in $x^*$:
$$\text{Im} F' (x^*) = 0. \quad (4)$$
A2) operator F is 2-regular in $x^*$:
$$\text{Im} F' (x^*) h = \mathbb{R}^m \quad \text{for} \ h \in K_2(x^*), \ h \neq 0. \quad (5)$$
A3) $$\text{Ker} F' (x^*) \neq \{0\}. \quad (6)$$

If F satisfies A1 in $x^*$, then
$$K_2(x^*) = \text{Ker}^2 P^\perp F' (x^*) = \{ h \in \mathbb{R}^n : F' (x^*)[h]^2 = 0 \}. \quad (7)$$

In [1] it was proved, that if $n=m$, then the sequence
$$x_{k+1} = x_k - \left( \hat{F}' (x_k) + P_k^\perp F' (x_k) h_k \right)^{-1} \cdot \left( F (x_k) + P_k^\perp F' (x_k) h_k \right), \quad (8)$$
where
$$P_k^\perp$$ - denotes orthogonal projection on $\left( \text{Im} \hat{F}' (x_k) \right)^\perp$ in $\mathbb{R}^n$,
$$h_k \in \text{Ker} \hat{F}' (x_k), \quad \| h_k \| = 1$$
converges Q-quadratically to $x^*$.

The matrices $\hat{F}' (x_k)$ obtained from $F' (x_k)$ by replacing all elements, whose absolute values do not increase $\nu>0$, by zero, where $\nu = \nu_k = \| F (x_k) \|^{(1-\alpha)/2}, \ 0<\alpha<1$.

In the case $n = m+1$ the operator
$$\left\{ \hat{F}' (x_k) + P_k^\perp F' (x_k) h_k \right\}^{-1}$$
in method (8) is replaced by the operator
$$\left[ \hat{F}' (x_k) + P_k^\perp F' (x_k) h_k \right]^\top \quad (9)$$
and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined ($n>m$) and degenerated in $x^*$. 

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2. Extending of the system of equation

Now we construct the operator $\Phi : R^n \rightarrow R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*)=0$ [2].

Assume A4) Let $F(x)=[f_1(x), f_2(x), \ldots, f_m(x)]^T$, $n>m$ is two continuously differentiable in some neighbourhood $U \subset R^n$ of the point $x^*$.

Denote:

$$H=\text{lin}\{h\} \quad \text{for} \quad h \in \text{Ker}^2 F'(x^*) \quad h \neq 0.$$  

$P = P_{H^\perp}$ denotes the orthogonal projection $R^n$ on $H^\perp$

$$f_i'(x) = P(F(x))$$  

for $i=1,2,\ldots,m$.

For each system of indices $i_1, i_2, \ldots, i_{n-m-1} \subset \{1, 2, \ldots, m\}$ and vectors $h_1, h_2, \ldots, h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x) h \\ \varphi(x) \end{bmatrix},$$  

where

$$\varphi(x) : R^n \rightarrow R^r, \quad r=n-m-1,$$

$$\varphi(x) = PF'(x) h^r, \quad h^r = [h_1, h_2, \ldots, h_r]^T,$$

$$\varphi(x) = M.$$  

(11)

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[\Phi'(x_k)\right]^+ \Phi(x_k), \quad k=0,1,2,\ldots$$  

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \left\{B_k\right\}^+ \Phi(x_k).$$  

(13)

The operator $\Phi'$ will be approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k.$$  

(14)

We propose matrices $B_k$ which satisfy the secant equation:

$$B_{k+1} s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for} \quad k=0,1,2,\ldots$$  

(15)

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:
\[ B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \] for \( k=0,1,2,... \) \hspace{1cm} (16)

where
\[ r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \] \hspace{1cm} (17)

We will prove for this method:

**Q-linear convergence** to \( x^* \) i.e. there exists \( q \in (0,1) \) such that
\[ \left\| x_{k+1} - x^* \right\| \leq q \left\| x_k - x^* \right\| \] for \( k = 0,1,2,... \) \hspace{1cm} (18)

and next **Q-superlinear convergence** to \( x^* \), i.e.:
\[ \lim_{k \to \infty} \frac{\left\| x_{k+1} - x^* \right\|}{\left\| x_k - x^* \right\|} = 0. \hspace{1cm} (19) \]

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator \( F'(x^*) \) is nonsingular.

**Theorem 1** (The Bounded Deterioration Theorem)

Let \( F \) satisfies the assumptions A1-A4. If exist constants \( q_1 \geq 0 \) and \( q_2 \geq 0 \) such that matrices \( \{B_k\} \) satisfy the inequality:
\[ \left\| B_{k+1} - \Phi'(x^*) \right\| \leq (1 + q_1 r_k) \left\| B_k - \Phi'(x^*) \right\| + q_2 r_k, \] \hspace{1cm} (20)

then there are constants \( \varepsilon > 0 \) i \( \delta > 0 \) such that if
\[ \left\| x_0 - x^* \right\| \leq \varepsilon \hspace{0.5cm} \text{and} \hspace{0.5cm} \left\| B_0 - \Phi'(x^*) \right\| \leq \delta, \]

then the sequence
\[ x_{k+1} = x_k - B_k^* \Phi(x_k) \]
converges **Q-linearly** to \( x^* \).

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

**Theorem 2** (Linear convergence)

Let \( F \) satisfies the assumptions A1-A4. Then the method
\[ x_{k+1} = x_k - \{B_k\}^* \cdot \Phi(x_k), \]

\[ B_{k+1} = B_k - \left\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \right\} s_k^T \]
locally and **Q-linearly** converges to \( x^* \).

**Proof.**

To prove the Theorem we should prove the inequality (20) from Theorem 1.

Now we notice:
Theorem 3 (Q-superlinear convergence)
Let F satisfies the assumptions A1-A4 and the sequence
\[ x_{k+1} = x_k - \{B_k\}^{-1} \cdot \Phi(x_k), \]
\[ B_{k+1} = B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k} \]
linearly converges to \( x^* \). Then the sequence \( \{x_k\} \) Q-superlinearly converges to \( x^* \).

Proof.
Matrices \( B_k \) satisfy secant equation (15), so
\[ B_{k+1} = P_{L_k}^{-1} B_k \] (21)
where
\[ L_k = \left\{ X : X s_k = y_k, \text{ where } y_k = \Phi'(x_{k+1}) - \Phi'(x_k) \right\} \] (22)

Denote
\[ H_k = H(x_k, x_{k+1}) = \frac{1}{0} \Phi'(x_k + t(x_{k+1} - x_k)) dt. \]
We have \( H_k \in L_k \) [4].
From (21) and [3] it follows:
\[ \| B_{k+1} - B_k \|^2 + \| B_{k+1} - H_k \|^2 = \| B_k - H_k \|^2, \quad \text{for } i = 0, 1, 2, \ldots. \]
By lemma 2 [5] we get
\[ \sum_{k=1}^{\infty} \| B_{k+1} - B_k \|^2 < \infty, \]
thus we obtain
\[ \| B_{k+1} - B_k \| \to 0. \]
This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. □

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of \( F^{(n)}(x_k) \).

References