Newton-like method for singular 2-regular system of nonlinear equations

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Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

1. Introduction

Let $F : D \subset R^n \rightarrow R^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^* \in D$ of the equation

$$F(x) = 0.$$ (1)

Definition 1

A linear operator $\Psi_2(h) : R^n \rightarrow R^m$, $h \in R^n$, is called 2-factor operator, if

$$\Psi_2(h) = F'(x^*) + P^\perp F^\perp(x^*) h,$$ (2)

where

$P^\perp$ denotes the orthogonal projection on $\left(\text{Im} F'(x)\right)^\perp$ in $R^n$ [1].

Definition 2

Operator $F$ is called 2-regular in $x^*$ on the element $h \in R^n$, $h \neq 0$, if the operator $\Psi_2(h)$ has the property:

$$\text{Im} \Psi_2(h) = R^m.$$ 

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Definition 3
Operator F is called 2-regular in $x^*$, if F is 2-regular on the set $K_2(x^*)\{0\}$, where

$$K_2(x^*) = \text{Ker}F^*(x^*) \cap \text{Ker}^2 P^+ F^*(x^*),$$

and

$$\text{Ker}^2 P^+ F^*(x^*) = \left\{ h \in R^n : P^+ F^*(x^*)[h]^2 = 0 \right\}.$$  

We need the following assumption on F:
A1) completely degenerated in $x^*$:

$$\text{Im} F^*(x^*) = 0.$$  

A2) operator F is 2-regular in $x^*$:

$$\text{Im} F^*(x^*) h = R^m \text{ for } h \in K_2(x^*), h \neq 0.$$  

A3) $\text{Ker} F^*(x^*) \neq \{0\}.$

If F satisfies A1 in $x^*$, then

$$K_2(x^*) = \text{Ker}^2 F^*(x^*) = \left\{ h \in R^n : F^*(x^*)[h]^2 = 0 \right\}.$$  

In [1] it was proved, that if $n=m$, then the sequence

$$x_{k+1} = x_k - \left\{ \hat{F}^*(x_k) + P^+_k F^*(x_k) h_k \right\}^{-1} \cdot \left\{ F(x_k) + P^+_k F^*(x_k) h_k \right\},$$  

where

$$P^+_k - \text{denotes orthogonal projection on } \left( \text{Im} \hat{F}^*(x_k) \right)^\perp \text{ in } R^n,$$

$$h_k \in \text{Ker} \hat{F}^*(x_k), \quad \| h_k \| = 1$$

converges Q-quadratically to $x^*$.

The matrices $\hat{F}^*(x_k)$ obtained from $F^*(x_k)$ by replacing all elements, whose absolute values do not increase $\nu>0$, by zero, where $\nu = \nu_k = \| F(x_k) \|^{(1-\alpha)/2}, 0<\alpha<1$.

In the case $n = m+1$ the operator

$$\left\{ \hat{F}^*(x_k) + P^+_k F^*(x_k) h_k \right\}^{-1}$$

in method (8) is replaced by the operator

$$\left[ \hat{F}^*(x_k) + P^+_k F^*(x_k) h_k \right]^+$$  

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined ($n>m$) and degenerated in $x^*$. 

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2. Extending of the system of equation

Now we construct the operator $\Phi : R^n \to R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*)=0$ \[2\].

Assume

A4) Let $F(x)=[f_1(x), f_2(x), \ldots, f_m(x)]^T$, $n>m$ is two continuously differentiable in some neighbourhood $U \subset R^n$ of the point $x^*$.

Denote:

$$H=\text{lin}\{h\} \quad \text{for} \quad h \in \text{Ker} F^2(x^*), \quad h \neq 0.$$  

$$P = P_{H^\perp}$$ denotes the orthogonal projection $R^n$ on $H^\perp$.

$$f'_i(x) = P(f'_i(x))^T \quad \text{for} \quad i=1,2,\ldots,m.$$  

For each system of indices $i_1, i_2, \ldots, i_{n-m-1} \subset \{1, 2, \ldots, m\}$ and vectors $h_1, h_2, \ldots, h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x) h \\ \varphi(x) \end{bmatrix},$$  

where

$$\varphi(x) : R^n \to R^r, \quad r=n-m-1,$$

$$\varphi(x) = PF'(x) h, \quad h \in \text{Ker} F^2(x^*) \subset [h_1, h_2, \ldots, h_r]^T,$$

$$\varphi(x) = \begin{bmatrix} f'_{i_1}(x) h_{i_1} \\ \vdots \\ f'_{i_{n-m-1}}(x) h_{i_{n-m-1}} \end{bmatrix}.$$  

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[ \Phi(x_k) \right]^T \cdot \Phi(x_k), \quad k=0,1,2,\ldots$$  

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \left[ B_k \right]^+ \cdot \Phi(x_k).$$  

The operator $\Phi$ will be approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k.$$  

We propose matrices $B_k$ which satisfy the secant equation:

$$B_{k+1} s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for} \quad k=0,1,2,\ldots$$  

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:
\[ B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \] for \( k=0,1,2,\ldots \) \hspace{1cm} (16)

where
\[ r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \] \hspace{1cm} (17)

We will prove for this method:

**Q-linear convergence** to \( x^* \) i.e. there exists \( q \in (0,1) \) such that
\[ \| x_{k+1} - x^* \| \leq q \| x_k - x^* \| \] for \( k = 0,1,2,\ldots \) \hspace{1cm} (18)

and next **Q-superlinear convergence** to \( x^* \), i.e.:
\[ \lim_{k \to \infty} \frac{\| x_{k+1} - x^* \|}{\| x_k - x^* \|} = 0. \] \hspace{1cm} (19)

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator \( F^{-1}(x^*) \) is nonsingular.

**Theorem 1** (The Bounded Deterioration Theorem)

Let \( F \) satisfies the assumptions A1-A4. If exist constants \( q_1 \geq 0 \) and \( q_2 \geq 0 \) such that matrices \( \{B_k\} \) satisfy the inequality:
\[ \left\| B_{k+1} - \Phi\left(x^*\right) \right\| \leq (1 + q_1 r_k) \left\| B_k - \Phi\left(x^*\right) \right\| + q_2 r_k, \] \hspace{1cm} (20)

then there are constants \( \varepsilon > 0 \) and \( \delta > 0 \) such, that if
\[ \| x_0 - x^* \| \leq \varepsilon \quad \text{and} \quad \| B_0 - \Phi'\left(x^*\right) \| \leq \delta, \]

then the sequence
\[ x_{k+1} = x_k - B_k^{-1} \Phi(x_k) \]
converges Q-linearly to \( x^* \).

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

**Theorem 2** (Linear convergence)

Let \( F \) satisfies the assumptions A1-A4. Then the method
\[ x_{k+1} = x_k - \{B_k\}^+ \cdot \Phi(x_k), \]
\[ B_{k+1} = B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k} \]
locally and Q-linearly converges to \( x^* \).

**Proof.**

To prove the Theorem we should prove the inequality (20) from Theorem 1.

Now we notice:

\[ k \]
where $c_1 > 0$, $c_2 > 0$, $q_1 > 0$, $q_2 > 0$, $r_k = \max \{ \|x_{k+1} - x^*\|, \|x_k - x^*\| \}$. 

\textbf{Theorem 3}  \hspace{1em} \textbf{(Q-superlinear convergence)}

Let $F$ satisfies the assumptions A1-A4 and the sequence

\[ x_{k+1} = x_k - \left\{ B_k \right\}^{-1} \cdot \Phi'(x_k) , \]

\[ B_{k+1} = B_k - \frac{\left\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \right\} s_k^T}{s_k^T s_k} \]

linearly converges to $x^*$. Then the sequence $\{x_k\}$ Q-superlinearly converges to $x^*$.

\textbf{Proof.}

Matrices $B_k$ satisfy secant equation (15), so

\[ B_{k+1} = P_{L_k}^{-1} B_k \]

where

\[ L_k = \{ X : X s_k = y_k , \text{ where } y_k = \Phi'(x_{k+1}) - \Phi'(x_k) \} \]

Denote

\[ H_k = H(x_k , x_{k+1}) = \int_0^1 \Phi'(x_k + t(x_{k+1} - x_k)) dt . \]

We have $H_k \in L_k$ [4].
From (21) and [3] it follows:
\[ \| B_{k+1} - B_k \|^2 + \| B_{k+1} - H_k \|^2 = \| B_k - H_k \|^2, \text{ for } i = 0, 1, 2, \ldots \]

By lemma 2 [5] we get \( \sum_{k=1}^{\infty} \| B_{k+1} - B_k \|^2 < \infty \), thus we obtain
\[ \| B_{k+1} - B_k \| \to 0. \]

This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. \( \square \)

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of \( F''(x_k) \).

References