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Newton-like method for singular 2-regular system of nonlinear equations

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Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear converegence for the nonlinear operators is developed.

1. Introduction

Let $F : D \subset R^n \to R^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^* \in D$ of the equation

$$F(x) = 0. \tag{1}$$

Definition 1

A linear operator $\Psi_2(h) : R^n \to R^m$, $h \in R^n$ is called 2-factoroperator, if

$$\Psi_2(h) = F'(x^*) + P^{\perp} F''(x^*) h, \tag{2}$$

where

$P^{\perp}$ - denotes the orthogonal projection on $(\operatorname{Im} F'(x))^{\perp}$ in R^n [1].

Definition 2

Operator F is called 2-regular in x^* on the element $h \in R^n$, $h \neq 0$, if the operator $\Psi_2(h)$ has the property:

$$\operatorname{Im} \Psi_2(h) = R^m.$$

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Definition 3

Operator F is called 2-regular in x^*, if F is 2-regular on the set $K_2(x^*)\backslash\{0\}$, where

$$K_2\left(x^*\right)=KerF'\left(x^*\right)\cap Ker^2P^{\perp}F''\left(x^*\right), \tag{3}$$

$$Ker^2P^{\perp}F''\left(x^*\right)=\left\{h\in R^n : P^{\perp}F''\left(x^*\right)[h]^2=0\right\}.$$

We need the following assumption on F:

A1) completely degenerated in x^*:

$$\operatorname{Im} F'\left(x^*\right)=0. \tag{4}$$

A2) operator F is 2-regular in x^*:

$$\operatorname{Im} F''\left(x^*\right)h=R^m \quad \text{for } h\in K_2(x^*),\ h\neq 0. \tag{5}$$

A3)
$$KerF''\left(x^*\right)\neq\{0\}. \tag{6}$$

If F satisfies A1 in x^*, then

$$K_2\left(x^*\right)=Ker^2F''\left(x^*\right)=\left\{h\in R^n : F''\left(x^*\right)[h]^2=0\right\}. \tag{7}$$

In [1] it was proved, that if n=m, then the sequence

$$x_{k+1}=x_k-\left\{\hat{F}'\left(x_k\right)+P_k^{\perp}F''\left(x_k\right)h_k\right\}^{-1}\cdot\left\{F\left(x_k\right)+P_k^{\perp}F'\left(x_k\right)h_k\right\}, \tag{8}$$

where

$P_k^{\perp}$ - denotes orthogonal projection on $\left(\operatorname{Im}\hat{F}'\left(x_k\right)\right)^{\perp}$ *in* R^n,

$$h_k\in Ker\hat{F}'\left(x_k\right),\quad \left\|h_k\right\|=1$$

converges Q-quadratically to x^*.

The matrices $\hat{F}'\left(x_k\right)$ obtained from $F'\left(x_k\right)$ by replacing all elements, whose absolute values do not increase ν>0, by zero, where $n=n_k=\left\|F\left(x_k\right)\right\|^{(1-a)/2}$, 0<α<1.

In the case n = m+1 the operator

$$\left\{\hat{F}'\left(x_k\right)+P_k^{\perp}F''\left(x_k\right)h_k\right\}^{-1}$$

in method (8) is replaced by the operator

$$\left[\hat{F}'\left(x_k\right)+P_k^{\perp}F''\left(x_k\right)h_k\right]^{+} \tag{9}$$

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined (n>m) and degenerated in x^*.

2. Extending of the system of equation

Now we construct the operator $\Phi : R^n \to R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*)=0$ [2].

Assume

A4) Let $F(x)=[f_1(x), f_2(x), ..., f_m(x)]^T$, n>m is two continuously differentiable in some neighbourhood $U\subset R^n$ of the point x^*.

Denote:

$$H=\mathrm{lin}\{h\} \qquad \text{for } h\in Ker^2 F''(x^*),\ h\neq 0.$$

$P = P_{H^\perp}$ denotes the orthogonal projection R^n on $H^\perp$

$$\tilde{f}_i'(x) = P\left(f_i'(x)\right)^T \text{ for } i=1,2,...,m.$$

For each system of indices $i_1, i_2, ..., i_{n-m-1} \subset \{1, 2, ..., m\}$ and vectors $h_1, h_2, ..., h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x)h \\ j(x) \end{bmatrix}, \tag{10}$$

where

$$j(x) : R^n \to R^r, \qquad r=n-m-1,$$

$$j(x) = PF'(x)\tilde{h}, \quad \tilde{h} = [h_1, h_2, ..., h_r]^T,$$

$$j(x) = \begin{bmatrix} \tilde{f}_{i_1}'(x)h_1 \\ \vdots \\ \tilde{f}_{i_r}'(x)h_r \end{bmatrix}. \tag{11}$$

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[\Phi'(x_k)\right]^+ \cdot \Phi(x_k), \qquad k=0,1,2,.... \tag{12}$$

quadraticaly converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \{B_k\}^+ \cdot \Phi(x_k). \tag{13}$$

The operator Φ' will by approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k. \tag{14}$$

We propose matrices B_k which satisfy the secant equation:

$$B_{k+1}s_k = \Phi(x_{k+1}) - \Phi(x_k) \qquad \text{for } k=0,1,2,... \tag{15}$$

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:

$$B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \qquad \text{for } k=0,1,2,... \tag{16}$$

where

$$r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \tag{17}$$

We will prove for this method:

Q-linear convergence to x^* i.e. there exists $q \in (0,1)$ such, that

$$\left\| x_{k+1} - x^* \right\| \le q \left\| x_k - x^* \right\| \qquad \text{for } k = 0,1,2,... \tag{18}$$

and next Q-superlinear convergence to x^*, i.e.:

$$\lim_{k\to\infty} \frac{\left\| x_{k+1} - x^* \right\|}{\left\| x_k - x^* \right\|} = 0\,. \tag{19}$$

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator $F'(x^*)$ is nonsingular.

<u>Theorem 1</u> (The Bounded Deterioration Theorem)

Let F satisfies the assumptions A1-A4. If exist constants $q_1 \ge 0$ and $q_2 \ge 0$ such that matrices $\{B_k\}$ satisfy the inequality:

$$\left\| B_{k+1} - \Phi'\left(x^*\right) \right\| \le \left(1 + q_1 r_k\right) \left\| B_k - \Phi'\left(x^*\right) \right\| + q_2 r_k\,, \tag{20}$$

then there are constants e >0 i d >0 such, that if

$$\|x_0 - x^*\| \le e \quad \text{and} \quad \| B_0 - \Phi'(x^*) \| \le d\,,$$

then the sequence

$$x_{k+1} = x_k - B_k^+ \Phi\left(x_k\right)$$

converges Q-linearly to x^*.

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

<u>Theorem 2</u> (Linear convergence)

Let F satisfies the assmuptions A1-A4. Then the method

$$x_{k+1} = x_k - \left\{B_k\right\}^+ \cdot \Phi\left(x_k\right),$$

$$B_{k+1} = B_k - \frac{\left\{\Phi\left(x_{k+1}\right) - \Phi\left(x_k\right) - B_k s_k\right\} s_k^T}{s_k^T s_k}$$

locally and Q-linearly converges to x^*.

<u>Proof.</u>

To prove the Theorem we should prove the inequality (20) from Theorem 1. Now we notice:

$$\left\|B_{k+1}-\Phi'\left(x^*\right)\right\|=\left\|B_k-\frac{\left\{\Phi\left(x_{k+1}\right)-\Phi\left(x_k\right)-B_ks_k\right\}s_k^T}{s_k^Ts_k}-\Phi'\left(x^*\right)\right\|\le$$

$$\le\left\|B_k-\Phi'\left(x^*\right)\right\|+\left\|\frac{\left\{\Phi\left(x_{k+1}\right)-\Phi\left(x_k\right)-B_ks_k\right\}s_k^T}{s_k^Ts_k}\right\|\le\left\|B_k-\Phi'\left(x^*\right)\right\|+$$

$$+\left\|\frac{\left\{\Phi\left(x_{k+1}\right)-\Phi\left(x_k\right)-\Phi'\left(x^*\right)s_k+\Phi'\left(x^*\right)s_k-B_ks_k\right\}s_k^T}{s_k^Ts_k}\right\|\le\left\|B_k-\Phi'\left(x^*\right)\right\|+$$

$$+\left\|\frac{(\Phi(x_{k+1})-\Phi'(x^*)(x_{k+1}-x^*))s_k^T}{s_k^Ts_k}\right\|+\left\|\frac{(\Phi(x_k)-\Phi'(x^*)(x_k-x^*))s_k^T}{s_k^Ts_k}\right\|+$$

$$+\left\|\frac{(\Phi'(x^*)-B_k)s_ks_k^T}{s_k^Ts_k}\right\|\le\left\|\Phi'(x^*)-B_k\right\|(1+q_1r_k)+c_1\frac{\left\|x_{k+1}-x^*\right\|^2\left\|s_k\right\|}{\left\|s_k^Ts_k\right\|}+$$

$$+c_2\frac{\left\|x_k-x^*\right\|^2\left\|s_k\right\|}{\left\|s_k^Ts_k\right\|}\le\left\|\Phi'\left(x^*\right)-B_k\right\|\left(1+q_1r_k\right)+q_2r_k,$$

where $c_1>0$, $c_2>0$, $q_1>0$, $q_2>0$, $r_k=\max\{\|x_{k+1}-x^*\|, \|x_k-x^*\|\}$. $\square$.

<u>Theorem 3</u> (Q-superlinear convergence)

Let F satisfies the assmuptions A1-A4 and the sequence

$$x_{k+1}=x_k-\left\{B_k\right\}^{-1}\cdot\Phi\left(x_k\right),$$

$$B_{k+1}=B_k-\frac{\left\{\Phi\left(x_{k+1}\right)-\Phi\left(x_k\right)-B_ks_k\right\}s_k^T}{s_k^Ts_k}$$

linearly converges to x^*. Then the sequence $\{x_k\}$ Q-superlinearly converges to x^*.

Proof.

Matrices B_k satisfy secant equation (15), so

$$B_{k+1}=P_{L_k}^{\perp}B_k \tag{21}$$

where

$$L_k=\left\{X: Xs_k=y_k,\ \text{where } y_k=\Phi'\left(x_{k+1}\right)-\Phi'\left(x_k\right)\right\} \tag{22}$$

Denote

$$H_k=H\left(x_k,x_{k+1}\right)=\int_0^1\Phi'\Big(x_k+t\left(x_{k+1}-x_k\right)\Big)dt\,.$$

We have $H_k\in L_k$ [4].

From (21) and [3] it follows:

$$\left\|B_{k+1}-B_k\right\|^2+\left\|B_{k+1}-H_k\right\|^2=\left\|B_k-H_k\right\|^2, \quad \text{for i = 0, 1, 2, ...}$$

By lemma 2 [5] we get $\sum_{k=1}^{\infty}\left\|B_{k+1}-B_k\right\|^2<\infty$, thus we obtain

$$\| B_{k+1} - B_k \| \to 0.$$

This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. □

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of $F''(x_k)$.

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