Newton-like method for singular 2-regular system of nonlinear equations

Stanisław Grzegórski*, Edyta Łukasik

Department of Computer Science, Lublin University of Technology, Nadbystrzycka 36b, 20-618 Lublin, Poland

Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

1. Introduction

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^* \in D$ of the equation

$$ F(x) = 0. \quad (1) $$

Definition 1

A linear operator $\Psi_2(h) : \mathbb{R}^n \to \mathbb{R}^m$, $h \in \mathbb{R}^n$, is called 2-factor operator, if

$$ \Psi_2(h) = F'(x^*) + P^\perp F'(x^*) h, \quad (2) $$

where $P^\perp$ - denotes the orthogonal projection on $(\text{Im } F'(x))^\perp$ in $\mathbb{R}^n [1]$.

Definition 2

Operator $F$ is called 2-regular in $x^*$ on the element $h \in \mathbb{R}^n$, $h \neq 0$, if the operator $\Psi_2(h)$ has the property:

$$ \text{Im } \Psi_2(h) = \mathbb{R}^m.$$
Definition 3
Operator $F$ is called 2-regular in $x^*$, if $F$ is 2-regular on the set $K_2(x^*)\{0\}$, where

$$K_2(x^*) = \ker F^+ (x^*) \cap \ker F^+ (x^*)^\perp,$$

(3)

$$\ker F^+ (x^*) = \{ h \in \mathbb{R}^n : P^+ F^+ (x^*)[h]^2 = 0 \}.$$

We need the following assumption on $F$:
A1) completely degenerated in $x^*$:

$$\text{Im} F^+ (x^*) = 0.$$  

(4)
A2) operator $F$ is 2-regular in $x^*$:

$$\text{Im} F^+ (x^*) h = R^m \text{ for } h \in K_2(x^*), h \neq 0.$$  

(5)
A3) 

$$\ker F^+ (x^*) \neq \{0\}.$$

(6)

If $F$ satisfies A1 in $x^*$, then

$$K_2(x^*) = \ker F^+ (x^*) = \{ h \in \mathbb{R}^n : F^+ (x^*)[h]^2 = 0 \}.$$  

(7)

In [1] it was proved, that if $n=m$, then the sequence

$$x_{k+1} = x_k - \left\{ \hat{F}^+ (x_k) + P_k^+ F^+ (x_k) h_k \right\}^{-1} \cdot \left\{ F(x_k) + P_k^+ F^+ (x_k) h_k \right\},$$

(8)

where

$$P_k^+ \text{ - denotes orthogonal projection on } \left( \text{Im} \hat{F}^+ (x_k) \right)^\perp \text{ in } \mathbb{R}^n,$$

$$h_k \in \ker \hat{F}^+ (x_k), \quad \|h_k\| = 1$$

converges Q-quadratically to $x^*$.

The matrices $\hat{F}^+ (x_k)$ obtained from $F^+ (x_k)$ by replacing all elements, whose absolute values do not increase $\nu > 0$, by zero, where $\nu = \nu_k = \|F(x_k)\|^{(1-\alpha)/2}$, $0 < \alpha < 1$.

In the case $n = m+1$ the operator

$$\{ \hat{F}^+ (x_k) + P_k^+ F^+ (x_k) h_k \}^{-1}$$

in method (8) is replaced by the operator

$$[ \hat{F}^+ (x_k) + P_k^+ F^+ (x_k) h_k ]^+$$

(9)

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined ($n > m$) and degenerated in $x^*$. 

PDF created with FinePrint pdfFactory Pro trial version www.pdffactory.com
2. Extending of the system of equation

Now we construct the operator $\Phi : R^n \rightarrow R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*)=0$ [2]. Assume

A4) Let $F(x)=[f_1(x), f_2(x), ..., f_m(x)]^T$, $n>m$ is two continuously differentiable in some neighbourhood $U \subset R^n$ of the point $x^*$.

Denote:

$$H=\text{lin}\{h\} \quad \text{for} \quad h \in \text{Ker} F'(x^*), \quad h \neq 0.$$ 

$P = P^t_H$ denotes the orthogonal projection $R^n$ on $H^\perp$.

$$j_i^q(x) = P(f_i'(x))^T$$ for $i=1,2,...,m$.

For each system of indices $i_1, i_2, ..., i_{n-m-1} \subset \{1, 2, ..., m\}$ and vectors $h_1, h_2, ..., h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x)h \\ \varphi(x) \end{bmatrix}, \quad \text{(10)}$$

where

$$\varphi(x) : R^n \rightarrow R^r, \quad r=n-m-1,$$

$$\varphi(x) = PF'(x)h_i^t, \quad h_i^t = [h_1, h_2, ..., h_i]^T,$$

$$\varphi(x) = \begin{bmatrix} \varphi(x)_h \\ \varphi(x)_{h_i} \end{bmatrix}.$$ 

(11)

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - \left[ \Phi'(x_k) \right]^+ \cdot \Phi(x_k), \quad k=0,1,2,.... \quad (12)$$

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \left( B_k \right)^+ \cdot \Phi(x_k). \quad (13)$$

The operator $\Phi'$ will by approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k. \quad (14)$$

We propose matrices $B_k$ which satisfy the secant equation:

$$B_{k+1}s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for} \ k=0,1,2,... \quad (15)$$

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method.
where
\[ r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \]  
(17)

We will prove for this method:

- **Q-linear convergence** to \( x^* \) i.e. there exists \( q \in (0,1) \) such that

\[
\|x_{k+1} - x^*\| \leq q \|x_k - x^*\| \quad \text{for } k = 0, 1, 2, \ldots
\]  
(18)

and next **Q-superlinear convergence** to \( x^* \), i.e.:

\[
\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0.
\]  
(19)

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator \( F' (x^*) \) is nonsingular.

**Theorem 1** (The Bounded Deterioration Theorem)

Let \( F \) satisfies the assumptions A1-A4. If exist constants \( q_1 \geq 0 \) and \( q_2 \geq 0 \) such that matrices \( \{B_k\} \) satisfy the inequality:

\[
\left\| B_{k+1} - \Phi' (x^*) \right\| \leq (1 + q_1 r_k) \left\| B_k - \Phi' (x^*) \right\| + q_2 r_k,
\]  
(20)

then there are constants \( \varepsilon > 0 \) and \( \delta > 0 \) such, that if

\[
\|x_0 - x^*\| \leq \varepsilon \quad \text{and} \quad \|B_0 - \Phi' (x^*)\| \leq \delta,
\]

then the sequence

\[ x_{k+1} = x_k - B_k^{-1} \Phi (x_k) \]

converges Q-linearly to \( x^* \).

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

**Theorem 2** (Linear convergence)

Let \( F \) satisfies the assumptions A1-A4. Then the method

\[
x_{k+1} = x_k - \{B_k\}^+ \cdot \Phi (x_k),
\]

\[
B_{k+1} = B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k}
\]

locally and Q-linearly converges to \( x^* \).

**Proof.**

To prove the Theorem we should prove the inequality (20) from Theorem 1.

Now we notice:
\[
\|B_{k+1} - \Phi'(x^*)\| \leq \left\| B_k - \left\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \right\} s_k^T - \Phi'(x^*) \right\| \\
\leq \left\| B_k - \Phi'(x^*) \right\| + \left\| \left\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \right\} s_k^T \right\| \leq \left\| B_k - \Phi'(x^*) \right\| + \\
+ \left\| \left\{ \Phi(x_{k+1}) - \Phi(x_k) - \Phi'(x^*) s_k + \Phi'(x^*) s_k - B_k s_k \right\} s_k^T \right\| \leq \left\| B_k - \Phi'(x^*) \right\| + \\
+ \left\| (\Phi(x_{k+1}) - \Phi'(x^*) (x_k - x^*)) s_k^T \right\| + \left\| (\Phi(x_k) - \Phi'(x^*) (x_k - x^*)) s_k^T \right\| \\
+ \left\| (\Phi(x^*) - B_k) s_k^T \right\| \leq \left\| \Phi'(x^*) - B_k \right\| (1 + q_1 r_k) + c_1 \left\| x_{k+1} - x^* \right\| s_k^T \right\| + \\
+c_2 \left\| x_k - x^* \right\| s_k^T \right\| \leq \left\| \Phi'(x^*) - B_k \right\| (1 + q_1 r_k) + q_2 r_k,
\]

where \( c_1 > 0, c_2 > 0, q_1 > 0, q_2 > 0, r_k = \max\{\|x_{k+1} - x^*\|, \|x_k - x^*\|\}. \)

\[\square\]

**Theorem 3** (Q-superlinear convergence)

Let \( F \) satisfies the assumptions A1-A4 and the sequence
\[
x_{k+1} = x_k - \left\{ B_k \right\}^{-1} \cdot \Phi(x_k),
\]
\[
B_{k+1} = B_k - \left\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \right\} s_k^T
\]

linearly converges to \( x^* \). Then the sequence \( \{x_k\} \) Q-superlinearly converges to \( x^* \).

Proof.

Matrices \( B_k \) satisfy secant equation (15), so
\[
B_{k+1} = P_{L_k} B_k \tag{21}
\]

where
\[
L_k = \left\{ X : X s_k = y_k \right\}, \text{ where } y_k = \Phi'(x_{k+1}) - \Phi'(x_k)
\]

Denote
\[
H_k = H(x_k, x_{k+1}) = \int_0^1 \Phi'(x_k + t(x_{k+1} - x_k)) dt.
\]

We have \( H_k \in L_k \) [4].
From (21) and [3] it follows:
\[ \|B_{k+1} - B_k\|^2 + \|B_{k+1} - H_k\|^2 = \|B_k - H_k\|^2, \text{ for } i = 0, 1, 2, \ldots. \]
By lemma 2 [5] we get \( \sum_{k=1}^{\infty} \|B_{k+1} - B_k\|^2 < \infty \), thus we obtain
\[ \|B_{k+1} - B_k\| \to 0. \]
This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. \[\square\]

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of \( F''(x_k) \).

References