Newton-like method for singular 2-regular system of nonlinear equations

Stanisław Grzegórski*, Edyta Łukasik

Department of Computer Science, Lublin University of Technology, Nadbystrzycka 36b, 20-618 Lublin, Poland

Abstract
In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

1. Introduction
Let \( F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution \( x^* \in D \) of the equation
\[
F(x) = 0.
\] (1)

Definition 1
A linear operator \( \Psi_2(h) : \mathbb{R}^n \rightarrow \mathbb{R}^m \), \( h \in \mathbb{R}^n \) is called 2-factor operator, if
\[
\Psi_2(h) = F'(x^*) + P^\perp F(x^*) h,
\] (2)
where
\( P^\perp \) denotes the orthogonal projection on \((\text{Im } F'(x))^\perp\) in \( \mathbb{R}^n \) [1].

Definition 2
Operator \( F \) is called 2-regular in \( x^* \) on the element \( h \in \mathbb{R}^n, h \neq 0 \), if the operator \( \Psi_2(h) \) has the property:
\[
\text{Im } \Psi_2(h) = \mathbb{R}^m.
\]

* Corresponding author: e-mail address: grzeg@pluton.pol.lublin.pl
Definition 3
Operator F is called 2-regular in \( x^* \), if F is 2-regular on the set \( K_2(x^*)\{0\} \), where

\[
K_2(x^*) = \ker F^* (x^* ) \cap \ker P^\perp F^* (x^* ) ,
\]

\[
\ker P^\perp F^* (x^* ) = \{ h \in R^n : P^\perp F^* (x^* )[h]^2 = 0 \} .
\]

We need the following assumption on F:
A1) completely degenerated in \( x^* \):
\[
\text{Im} F^* (x^* ) = 0 .
\]

A2) operator F is 2-regular in \( x^* \):
\[
\text{Im} F^* (x^* ) h = R^m \text{ for } h \in K_2(x^*) , \ h \neq 0 .
\]

A3)
\[
\ker F^* (x^*) \neq \{0\} .
\]

If F satisfies A1 in \( x^* \), then
\[
K_2(x^*) = \ker F^* (x^*) = \{ h \in R^n : F^* (x^* )[h]^2 = 0 \} .
\]

In [1] it was proved, that if \( n = m \), then the sequence

\[
x_{k+1} = x_k - \left( \hat{F}^* (x_k) + P_k^{\perp} F^* (x_k) h_k \right)^{-1} \cdot \left( F (x_k) + P_k^{\perp} F^* (x_k) h_k \right),
\]

where

\[
P_k^{\perp} \text{ denotes orthogonal projection on } \left( \text{Im} \hat{F}^* (x_k) \right)^{\perp} \text{ in } R^n ,
\]

\[
h_k \in \ker \hat{F}^* (x_k) , \quad \| h_k \| = 1
\]

converges Q-quadratically to \( x^* \).

The matrices \( \hat{F}^* (x_k) \) obtained from \( F^* (x_k) \) by replacing all elements, whose absolute values do not increase \( v > 0 \), by zero, where \( v = v_k = \| F (x_k) \|^{(1-\alpha)/2} \), \( 0 < \alpha < 1 \).

In the case \( n = m+1 \) the operator

\[
\left( \hat{F}^* (x_k) + P_k^{\perp} F^* (x_k) h_k \right)^{-1}
\]

in method (8) is replaced by the operator

\[
\left[ \hat{F}^* (x_k) + P_k^{\perp} F^* (x_k) h_k \right]^+
\]

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined \( (n > m) \) and degenerated in \( x^* \).
2. Extending of the system of equation

Now we construct the operator $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*)=0$ [2].

Assume

A4) Let $F(x)=[f_1(x), f_2(x), ..., f_m(x)]^T$, $n>m$ is two continuously differentiable in some neighbourhood $U \subset \mathbb{R}^n$ of the point $x^*$.

Denote:

$H=\text{lin}\{h\}$ for $h \in \ker F'(x^*)$, $h \neq 0$.

$P = P_{H^\perp}$ denotes the orthogonal projection $\mathbb{R}^n$ on $H^\perp$.

For each system of indices $i_1, i_2, ..., i_{n-m-1} \subset \{1, 2, ..., m\}$ and vectors $h_1, h_2, ..., h_{n-m-1} \subset \mathbb{R}^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x) h_1 \\ \vdots \\ F'(x) h_{n-m-1} \\ \Phi(x) \end{bmatrix},$$

where

$$\phi(x) : \mathbb{R}^n \rightarrow \mathbb{R}^r, \quad r=n-m-1,$$

$$\phi(x) = P F'(x) \hat{h}, \quad \hat{h} \in [h_1, h_2, ..., h_r]^T,$$

$$\phi(x) = M$$

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - [\Phi'(x_k)]^- \cdot \Phi(x_k), \quad k=0,1,2,...$$

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence \{x_k\} is defined by:

$$x_{k+1} = x_k - (B_k)^+ \cdot \Phi(x_k).$$

The operator $\Phi'$ will be approximated by matrices \{B_k\}.

Let

$$s_k = x_{k+1} - x_k.$$

We propose matrices $B_k$ which satisfy the secant equation:

$$B_{k+1}s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for} \quad k=0,1,2,...$$

For example, to obtain the sequence \{B_k\} we can apply the Broyden method:
\[ B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \quad \text{for } k=0,1,2,... \] (16)

where
\[ r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \] (17)

We will prove for this method:

**Q-linear convergence** to \( x^* \), i.e. there exists \( q \in (0,1) \) such that
\[ \| x_{k+1} - x^* \| \leq q \| x_k - x^* \| \quad \text{for } k = 0,1,2,... \] (18)

and next **Q-superlinear convergence** to \( x^* \), i.e.:
\[ \lim_{k \to \infty} \frac{\| x_{k+1} - x^* \|}{\| x_k - x^* \|} = 0. \] (19)

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator \( F'(x^*) \) is nonsingular.

**Theorem 1** (The Bounded Deterioration Theorem)

Let \( F \) satisfies the assumptions A1-A4. If exist constants \( q_1 \geq 0 \) and \( q_2 \geq 0 \) such that matrices \( \{B_k\} \) satisfy the inequality:
\[ \left\| B_{k+1} - \Phi'(x^*) \right\| \leq (1 + q_1 r_k) \| B_k - \Phi'(x^*) \| + q_2 r_k, \] (20)

then there are constants \( \varepsilon > 0 \) and \( \delta > 0 \) such that if
\[ \| x_0 - x^* \| \leq \varepsilon \quad \text{and} \quad \| B_0 - \Phi'(x^*) \| \leq \delta, \]

then the sequence
\[ x_{k+1} = x_k - B_k^* \Phi(x_k) \]

converges Q-linearly to \( x^* \).

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

**Theorem 2** (Linear convergence)

Let \( F \) satisfies the assumptions A1-A4. Then the method
\[ x_{k+1} = x_k - \{ B_k \}^* \cdot \Phi(x_k), \]
\[ B_{k+1} = B_k - \frac{\{ \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k \} s_k^T}{s_k^T s_k} \]

locally and Q-linearly converges to \( x^* \).

**Proof.**

To prove the Theorem we should prove the inequality (20) from Theorem 1. Now we notice:
where $c_1>0$, $c_2>0$, $q_1>0$, $q_2>0$, $r_k = \max\{\|x_{k+1} - x^*\|, \|x_k - x^*\|\}$.

\begin{proof}

\end{proof}

\textbf{Theorem 3 (Q-superlinear convergence)}

Let $F$ satisfies the assumptions A1-A4 and the sequence

\[ x_{k+1} = x_k - B_k^{-1} \Phi'(x_k), \]

\[ B_{k+1} = B_k - \frac{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k}{s_k^T s_k} s_k^T \]

linearly converges to $x^*$. Then the sequence \{x_k\} Q-superlinearly converges to $x^*$.

\begin{proof}

Matrices $B_k$ satisfy secant equation (15), so

\[ B_{k+1} = P_{L_k} B_k \]  \quad (21)

where

\[ L_k = \{X : Xs_k = y_k, \text{ where } y_k = \Phi'(x_{k+1}) - \Phi'(x_k)\} \]  \quad (22)

Denote

\[ H_k = H(x_k, x_{k+1}) = \int_0^1 \Phi'(x_k + t(x_{k+1} - x_k)) \, dt. \]

We have $H_k \in L_k$ [4].
From (21) and [3] it follows:
\[ \left\| B_{k+1} - B_k \right\|^2 + \left\| B_{k+1} - H_k \right\|^2 = \left\| B_k - H_k \right\|^2, \quad \text{for } i = 0, 1, 2, \ldots. \]

By lemma 2 [5] we get \( \sum_{k=1}^{\infty} \left\| B_{k+1} - B_k \right\|^2 < \infty \), thus we obtain
\[ \left\| B_{k+1} - B_k \right\| \to 0. \]

This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. □

4. Summary
The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of \( F^{"}(x_k) \).

References