Universality classes of some probabilistic cellular automata

Norbert Sendra, Tomasz Gwizdała[*], Jerzy Czerbniak

Department of Solid State Physics, University of Łódź
ul. Pomorska 149/153, 90-236 Łódź, Poland

Abstract

The critical properties of one-dimensional, probabilistic cellular automata with two absorbing states are presented. Size dependent values of critical exponents related to order parameter and its susceptibility is analyzed and some inconsistencies in classification of this model into universality class are discussed.

Introduction

The idea of cellular automata has been known since the pioneering ideas of Ulam, von Neumann and Burks which appeared in the early 50-th of XX century [1-2]. CA represents very interesting model for numerical investigations because of relative simplicity caused especially by its features like well defined configuration space usually denoted simply by the array – like system of indices and most often a very narrow set of states which may be assigned to the nodes of the system. However, contrary to their simplicity cellular automata can reproduce the properties of a great number of theoretical approximations used in physics for many years. The Ising model discovered about 100 years ago, solved by Onsager for the two-dimensional case about 50 years ago but today still intensively studied for many different configurations of nodes and spins is a very good example.

Also outside physics cellular automata are used to model problems in such a wide range of areas like biology, sociology or transportation. One of the processes, which may be studied within the frame of CA approach, is the phenomenon of phase transition. The word "phase" may mean here as well the state which is well defined from the physical point of view (eg. ferro and paramagnet in the Ising model) as some global mathematical quantity calculated for the studied sample like the laminar flow or jam for traffic models.

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[*] Corresponding author: *e-mail address:* tomgwizd@uni.lodz.pl

In this work we will present some results concerning the phase transition in the mathematical model of one-dimensional CA with some characteristic properties.

2. Cellular Automata

Cellular automata are usually described as dynamical systems that are discrete in space and time, operating on a uniform, regular lattice and characterized by local interactions. In this description one should pay a special attention to the word "discrete", because the most important idea proposed by this model is the discretization of the problem. It concerns first of all the two variables, which are usually, and phenomenologically treated as continuous, it means both space coordinates and time. The discretization of space is realized as defining N-dimensional set of nodes usually built as a regular network. Thus the space can be easily described by the n-dimensional array. In one dimension it may be presented simply as a chain, for three dimensions eg. it reproduces the structure of simple cubic crystal. This network is the first of three entities needed to define the cellular automaton constructing the set of "cells" which we will index in the N-dimensional space as $i_1\, i_2 ...\, i_N$. The second one is the set of values, which can be assigned to the cells. By analogy to spin let us denote the value of $i_1\, i_2 ...\, i_N$-th cel as $s_{i1\, i2...\, iN}$. This value can be chosen only as the one from the unambiguously enumerated set of acceptable values. The third one describes the process of time evolution of the system. It is called the "rule". The rule is a function $s_{i1i2...\, iN}(t+1) = f(\{s_{\, j1j2...\, jN}(t)\})$. It means that the value of $i_1\, i_2 ...\, i_N$-th cell in the time step (t+1) depends on the values of other cells, usually surrounding it, in time step (t). Most frequently only the nearest neighbours are taken into account. It should be emphasized that there are used two types of nearest neighbours configurations. In the von Neumann's NN configuration only these cells for which exactly one index differs from the calculated cell index are considered. In Moore's NN configuration all cells having indices differing not more than one are used. Certainly for one-dimensional CA these two configurations are undistinguished, but for higher dimensions there are four or eight NN respectively (two dimensions) or 6 and 26 in 3D. It is clear that except for the simplest one-dimensional models the results will be strongly dependent on the type of neighbourhood considered. From the numerical point of view it leads also to significant differences in the time of performing the system evolution.

The typical, however not precise way of representing the CA is the (k,r) notation in which k represents the number of possible states of any cell and r is the number of neighbours taken into account when applying the rule.

There are different ways to classify cellular automata. Here we want to mention only one, very general classification dependent on the type of rule. If the state of cell is unambiguously determined by the configuration of

neighbouring cells the automaton is called deterministic. If it is also dependent on a value of some random variable we call it probabilistic. A very special type of automaton is the totalistic one, called also the voting automaton. The final state of cell doesn't on the configuration of NN, but depends on the sum of their spin states.

As the example of CA the one-dimensional let see at the one of the most intensively studied one-dimensional, (2,1), deterministic cellular automaton presented by Wolfram [3]. The two possible spin states are {0,1}. For each of 8 possible combinations of (i-1), i and i+1 spin states there must be defined a state which will be the result of application of the rule to the network. As an example consider the rule "90":

$$90 = (01011010)_2 \rightarrow \frac{111}{0}\frac{110}{1}\frac{101}{0}\frac{100}{1}\frac{011}{1}\frac{010}{0}\frac{001}{1}\frac{000}{0}.$$

Applying this rule to the network, initially characterized only by the one state 1 leads to one of typical patterns:
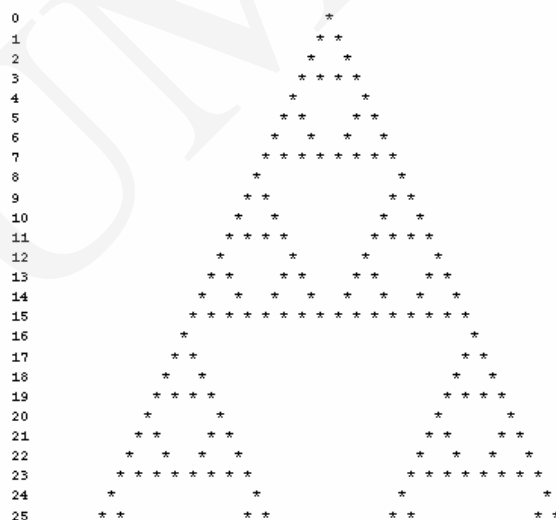

Fig. 1. Pattern produced by Wolfram's "90" automaton

3. Critical phenomena and universality classes

The phase transitions are the physical processes commonly observed. Its theoretical description is related to the specific behaviour of values characterizing the some physical properties near the transition point. If we observed eg. the dependence of magnetization of ferromagnetic sample on temperature we would observe that for low temperatures it is constant and suddenly very rapidly vanishes to zero at the Curie point. The similar behaviour

is also characteristic for other quantities characterized the ferro- to paramagnetic transition (susceptibility) as other phase transition processes like gas-liquid. Generally the power law may describe this functional dependence. The set of typically used values of exponents may be presented as follows (see [4]):

Table 1. The definitions of some critical exponents presented
for the ferromagnet to paramagnet phase transition

Exponent	definition
α	$c_B \propto ((\lvert T - T_{kr}\rvert / T_{kr})^{-\alpha} - 1)$
β	$m \propto (T_{kr} - T)^{\beta}$
γ	$\chi_T \propto \lvert T - T_{kr}\rvert^{-\gamma}$
δ	$m \propto B^{1/\delta}$
η	$G^{(2)}(r) \propto 1/r^{d-2+\eta}$
ν	$\xi \propto \lvert T - T_{kr}\rvert^{-\nu}$

However, the values in Tab.1 are given for magnetic case they can be easily adopted to other phenomena. In general the exponent β concerns the dependence of order parameter on some temperature-like parameter. The susceptibility χ, for the magnetic case defined as

$$c = \left(\frac{\partial m}{\partial B} \right)_{B=0}, \qquad (1)$$

may be understood as the derivative of order parameter over the external, ordering field. All the phase transitions have their own, specific set of values of critical exponents. The values of critical exponents for some well known theoretical models are presented in Tab.2.

Tab.2 Values of critical exponents for some well recognized theoretical models [4-6]

exponent	Ising 2D	Ising 3D	DP (1+1)	DP (2+1)
α	0 (log)	0.109		
β	1 / 8	0.3258	0.276	0.584
γ	7 / 4	1.2396	2.28	1.6
δ	15		0.159	0.451
η	1 / 4			
ν	1	0.6304	1.09/1.73	0.73/1.29

Here are presented the values for the simplest Ising models (two and three dimensional, for one dimensional, transition does not occur), and so-called directed percolation. The notation (n+1) means that we consider the n-dimensional model with time as an additional variable. The set of values of critical exponents constitutes the universality class in such a way that we can

attach the model studied into one of the existing classes comparing the values obtained for critical exponents with those characterizing a given class.

4. Model

In this work we took into account the model proposed by Bagnoli et al. [7-9]. It is defined as the totalistic, probabilistic (2,1) one-dimensional cellular automaton. In order to describe the rule, the function is defined:

$$q(i,t) = s(i-1,t) + s(i,t) + s(i+1,t). \tag{2}$$

Having the set of possible states equal to $\{0,1\}$, the set of values for q function is $\{0,1,2,3\}$ and we can construct the rule in the following way:

$$s(i,t+1) = \begin{cases} 0 & \text{if } q(i,t) = 0 \\ X_1 & \text{if } q(i,t) = 1 \\ X_2 & \text{if } q(i,t) = 2 \\ 1 & \text{if } q(i,t) = 3 \end{cases} \tag{3}$$

The variables X_1 and X_2 are the random Bernoulli variables equal to 1 ("active") with probability p_j and equal to 0 ("empty") with probability $1 - p_j$. So the probabilities of transitions are equal to:

$$P(0\,|\,000) = 1$$
$$P(1\,|\,001) = P(1\,|\,010) = P(1\,|\,100) = p_1$$
$$P(1\,|\,011) = P(1\,|\,110) = P(1\,|\,101) = p_2$$
$$P(1\,|\,111) = 1$$

The two absorbing states are characterized by this model, one of them corresponds to the situation when all neighbouring states are empty and the second one takes place when all are active.

5. Results and discussion

Our special attention was put onto the process of phase transition. In the model considered both stable points (absorbing states) may be treated as different phases if the density is considered as an order parameter. The phase diagrams are presented in Fig.2.

The plot shows that between two stable phases we can have both discontinuous and continuous phase transition. Going along the line $p_1+p_2=1$ from the upper left corner to the bicritical point characterized by the p_1 value about 0.55 a very sharp 0->1 transition may be observed but further there are two different phase transitions form 0 and to 1 state. In our calculations we use the parameter ε, which allows to determine the probabilities p_1 and p_2 in the way:

$$\begin{aligned} p_1 &= 0.5 * e \\ p_2 &= 0.5 * e + 0.5. \end{aligned} \tag{4}$$
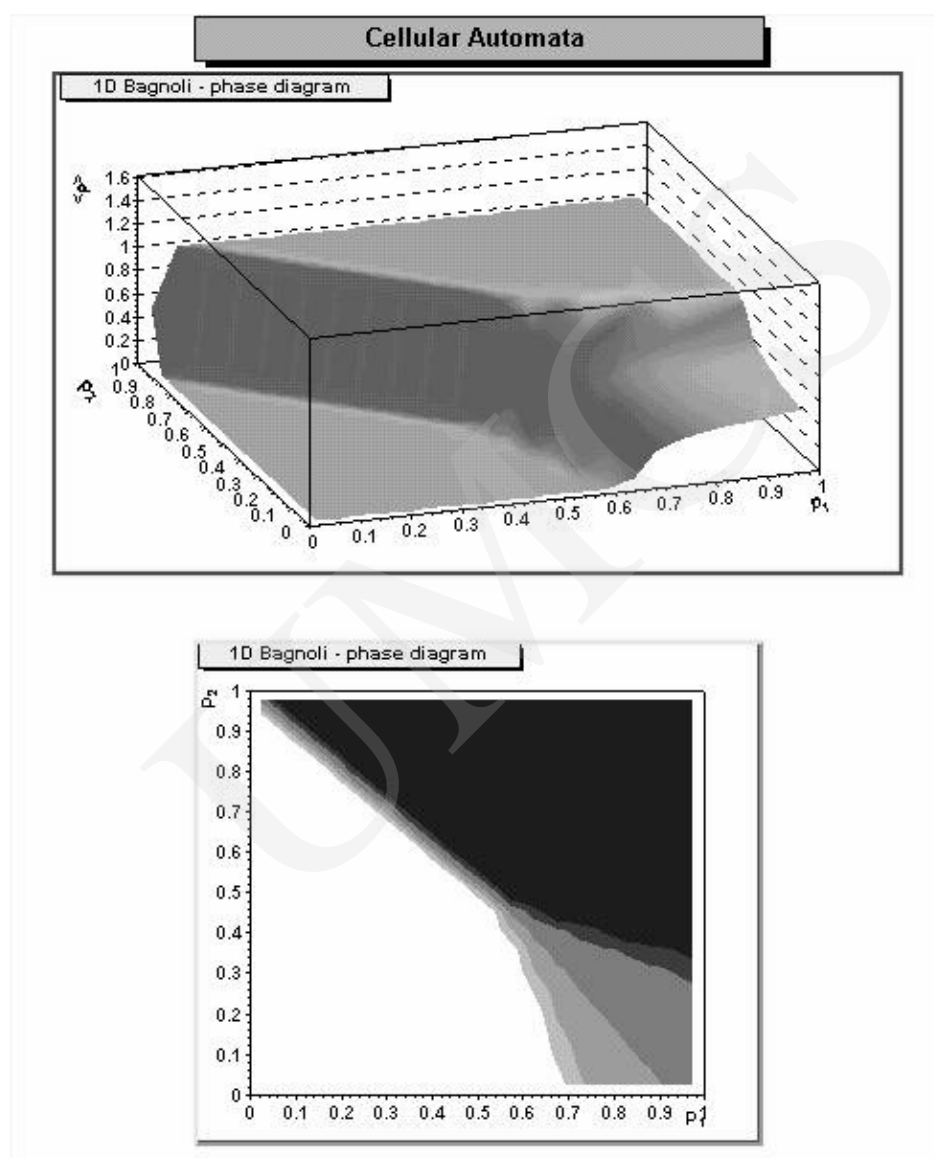
Fig. 2. Phase diagram for the Bagnoli automaton. The lower plot is the contour plot
with the levels distributed uniformly over (0.1) range

Geometrically it is simply the diagonal of the lower right square in the phase
diagram going from point (0.5,0.0) to (1.0,0.5). The two transitions mentioned
above may be clearly seen on the plot of density dependence on ε parameter.
The data presented in Fig.3 were obtained for the sample built of 100000 spins.
All further presented results were calculated for the two types of sample: the
smaller one containing one thousand cells (L=1k) and the bigger one as it was

written earlier having hundred thousand of spins (L=100k). We can assume that such a great sample may reproduce the behaviour of infinite automaton.
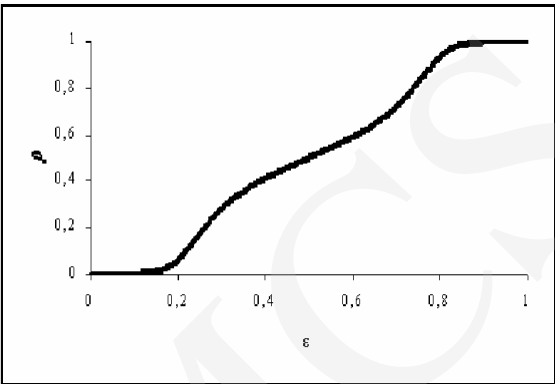

Fig.3. Density plot along the line described in the text

A very special feature of the model is the symmetry of both transitions from the absorbing states. Therefore we can consider more exactly only one of them and we choose the change of density from the zero state near $p1=0.6$ ($\varepsilon=0.2$). The crucial point is to find as well as it is possible the point of transition. In order to do this we use the plot of susceptibility

Because the point of transition should correspond to the maximum value of susceptibility it may be observed that this point is different for both sample sizes. The values are presented in Tab.3.

Table 3. Coordinates of phase transition point.

	L=1k	L=100k
p_1	0.635	0.628

It should be pointed out that the points listed above are not the points where the value of density begins to deviate from zero. It is the point, as it is described by the theory of critical phenomena, where the left and right-sided derivatives of order parameter are not equal. Certainly the choice of critical point has strong influence on the value of critical exponents. The log-log plot which makes it possible to determine the exponent is shown in fig.5

The values listed explicity on the plots differ from each other although this difference is almost not significant. The similar procedure may be applied to the plot of susceptibility. The results, critical exponents b and g as defined in tab.1. are presented in tab.4.

Table 4. Critical properties of Bagnoli automaton

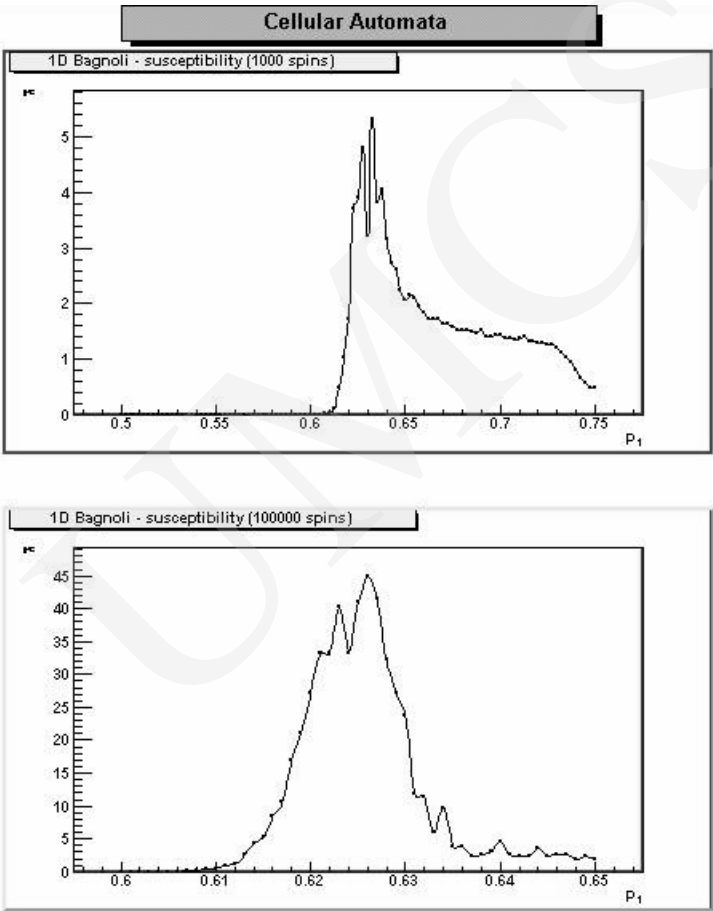|	L=1k	L=100k
β	0.145	0.125
γ	0.46	0.6

Fig. 4. The plot of susceptibility in the region of phase transition

These results show some interesting features especially when compared with some conclusions presented by the authors of model [9]. In their work they determined some other characteristic critical exponents like the average number of active sites $N(t)$, the survival probability $P(t)$ and the average square distance from the origin $R^2(t)$. Their analysis showed that cellular automaton presented belongs to the universality class of $(1+1)$ directed percolation. Our results lead, however, to different opinions. Comparing the values from tables 1 and 4 and taking into account the value of β parameter for the bigger sample one can say

that it reproduces with good accuracy the properties of two-dimensional Ising model. The problem is, however, with the γ exponent which is about two times smaller than the predicted one.
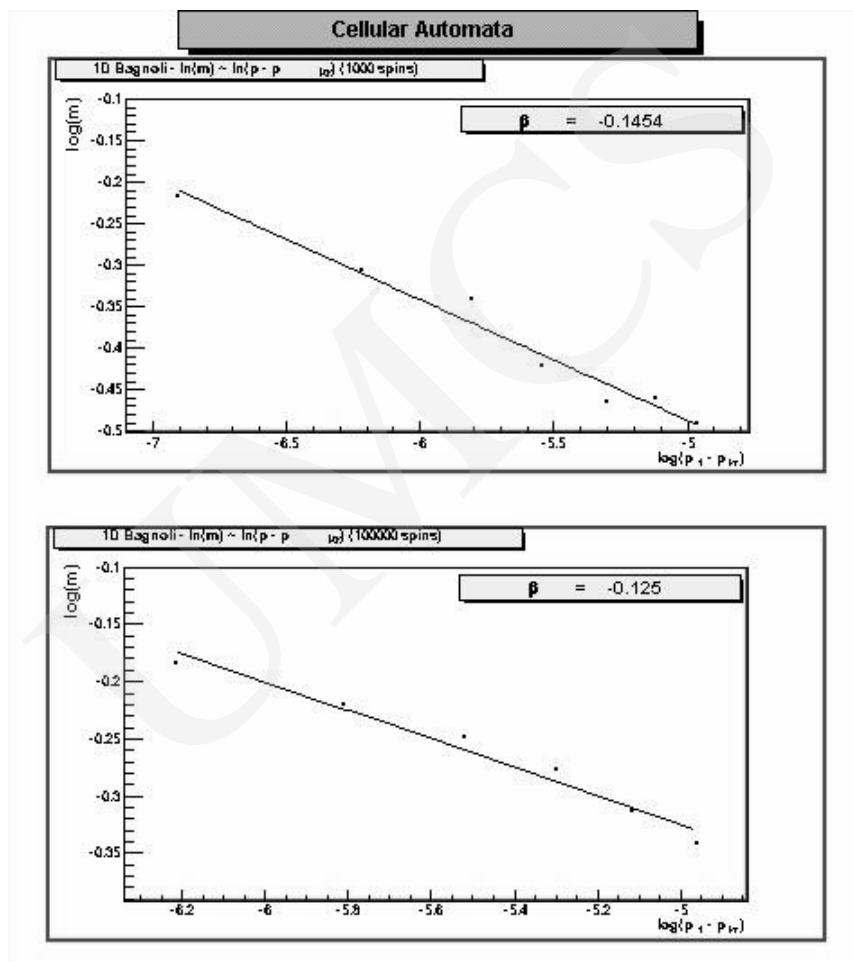

Fig. 5. Log-log plot of order parameter (density) in the region of phase transition

The results presented in our paper show some further needs and possibilities. First of all the other critical exponents should be calculated. The ν value describing the correlations may have a special role. Applying some equations connecting the values of different coefficients may give also the information about the correctness or errors in the fits made.

In the situation when the model is suitable to represent the Ising universality class we can think about reusing it in the Ising calculations by simple change of

lower absorbing state 0->(-1). The density will become then the sense of magnetization with the linear formula m=2(ρ-0.5).

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