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The four-color theorem and its consequences for the philosophy of mathematics

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Abstract

In the paper ways in which computers are applied in mathematics are considered. An example of mathematical truth which possesses only computer-assisted proof is the four-color theorem. Based on the example of this theorem we discuss some philosophy connected with admitting computer proofs in mathematics, in particular the status of mathematical knowledge as a pattern of science whose truths are known a priori.

1. The four-color theorem

In 1976 the four color problem, formulated over a hundred years earlier by Francis Guthrie, was solved. This problem was whether every map on the plane or sphere can be colored by no more than four colors in such a way that neighboring regions are never colored alike. (By regions we mean connected domains and we treat as neighboring such regions whose borders have more than one common point.)[1]

The proof, published in 1976 by Kenneth Appel and Wolfgang Haken [1], takes a meaningful advantage of computers. The fact that the proof used computers, hence an empirical element devide, implies that many mathematicians do not accept this proof, and never mention the four-color *theorem*, but only the four-color hypothesis.

The scheme of Appel and Haken's proof is simple. It is a proof by induction, which requires three cases. One of them is trivial, the second one includes two sub-cases, but the last one includes almost two thousands sub-cases and the checking of majority of them requires referring to computers.

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[1] If we assume that neighboring regions can have only one common point, it will be very easy to indicate a map, for example similar to birthday cake, whose coloring requires as many colors as regions.

In mathematics there are many theorems, whose proofs consist in checking cases, but the four-color theorem differs from them because:

– This theorem does not possess traditional mathematical proof, namely such a proof which does not require the usage of computers. Moreover two different proofs of this theorem are known today and both employ computer programs.
– Appel and Haken employed computer in the special way. They wrote two procedures, one of them was generating the elements of some set and the other was checking if these elements have a certain property. These two procedures cooperated in such a way that if the second procedure checked that some element does not possess required property, the other procedure was modified so that it would not produce sets including this element. So one can say that in this program there was an element of dialog between a machine (computer) and a man. Moreover, the program sometimes prompted new strategies to its authors:
  "It would out compound strategies based on all the tricks it had been "taught" and often these approaches were far more clever than those we would have tried. Thus it began to teach us things how to proceed that we never expected. In a sense it had surpassed its creators in some aspects of the "intellectual" as well as the mechanical parts of the task."[2]

The four-color theorem reveals philosophical problems connected with computer proofs. The situation is so special because in the case of other theorems proved by appealing to computers there was also a traditional proof. The four color problem is very easy to formulate and understand, but over a hundred years it remained a hypothesis, despite the fact that the problem was well known and considered by many mathematicians (professionals and amateurs) who tried to prove it. All attempts of solving this problems failed and finally computers enabled people to create the proof. A consciousness of unavoidability of computers in justification the four-color hypothesis, is the reason why this theorem was and still is so widely discussed.

2. The usage of computers in mathematics

Computers have been used in mathematics in many ways.

a) The oldest and still the most popular usage of computers are automatic calculations. The examples of such use are: testing the natural number is prime, finding roots of equations and solving integrals. In such applications computer can be treated as a more complicated calculator or abacus, which only enable man to shorten long/complicated calculations.

[2] Appel K., Haken W., *The Four-Color problem*, in: *Mathematics Today: Twelve Informal Essays*, L.A. Steen (Eds.), Springer-Verlag, New York-Heidelberg-Berlin, (1978) 153.

Computer carries out calculations in the similar way as a human mathematician, but faster and almost never makes mistakes.

However, we must remember that there are some limitations of computer counting. Real numbers are represented in computers with some errors. If some real number cannot be represented exactly, it is replaced by the closest number representable in computer. Such error of representation can cause further errors in the course of computations and finally a miscalculation. Every use of computers in order to solve problems of continuous mathematics, requires analysis of rounding errors and errors of representations. Without such analysis the obtained results can be interpreted mistakenly.

b) Computers are used in mathematics also in a more sophisticated way. They are used for example for carrying out experiments with mathematical objects and investigating their properties. Such possibility is often treated as an irrefutable argument for independent, objective existence of mathematical objects [2].

c) But the most controversial application of computers is proving theorems. In fact, this activity has the main influence on establishing new truths in mathematics and has been for ages kept to human mathematicians. Nowadays computers are very often incorporated in proving process in many ways.

First, computers check traditional (made by men) proofs. Proofs are coded in some formal language and then checked by special programs. Such application of computers does not establish new truths, but only enables us to check the existing ones. In the case of automatic proving the situation is different.

d) Automatic proving of theorems is closely connected with the idea of mechanization of reasoning. This idea dates from XVII/XVIII century, when G. W. Leibniz tried to create a formal language for all types of sciences and an universal method of calculation which would enable us to solve all problems. This method was called *calculus universalis* or *logica mathematica*.

The appearance of computers made it possible to realize practically of some existing algorithms of automatic proving.[3] We can divide automatically proved theorems into two groups: 1. theorems which possess traditional proofs and 2. theorems which are essentially new truths in mathematics. The existence of theorems of the second type leads to the problem of the role and place of computers in mathematics [3]. The same

[3] Nowadays, studies on mechanization of reasoning form a part of researches of the artificial intelligence.

problem appears in connection with, the so called computer-assisted proofs.

e) Computer-assisted proofs are more common in mathematics than automatic proofs. Computers are used in such proofs only in order to shorten long/difficult computations, but reasoning is carried out by a human mathematician. In this case a computer is only a tool in hands of a man. The proof of the four-color theorem belongs just to this category. As mentioned above, some mathematicians and philosophers do not accept such proofs and never call the four-color problem the *theorem*. So, we should consider what are the features of a mathematical proof and what is the difference between traditional proofs and computer-assisted ones.

3. Characteristics of mathematical proofs

The concept of a mathematical proof is commonly used in mathematics, but in spite of its intuitive evidence it is very hard to define it precisely. We think that a mathematician asked to define this concept would state similar, but not the same characteristics. Thomas Thymoczko in paper [4] makes an attempt to give the strict characteristics of a mathematical proof. He stresses three main features of it:

– Proofs are formalizable.

A proof, as defined in logic, is a finite sequence of formulas of a formal theory satisfying certain conditions. Most mathematicians and philosophers believe that any acceptable mathematical proof can be formalized. We can find an appropriate formal language and a theory in which the informal proof can be represented as a formal, strict one. Notice that a large majority of mathematical theorems does not possess such proofs. In the case of the four-color theorem we assume that there is a formal proof of this theorem in an appropriate theory of graphs, despite the fact, till nowadays nobody has written such a proof or even presented justification of its existence. The considered feature of proofs is not a feature which distinguishes computer proofs from classical ones.

– Proofs are surveyable.

This characteristics of proofs is more problematic than the first one. We must distinguish surveyability of the proof and its understanding. There are many traditional proofs which an average mathematician does not understand, but there is no proof which could not be looked over, reviewed and verified by a rational agent. The proof of the four-color theorem, like other computer proofs, does not possess this feature. No mathematician has surveyed the whole proof because there exists no printout of the results of the computer program. Moreover, even if such printout existed it would be so long that nobody could be able to read it even during his whole life. But in mathematics there are many proofs which have been read only by their authors and maybe by two other mathematicians. So, what is the difference between computer proofs and classical ones?

– Proofs are convincing.

Proofs must be convincing to mathematicians, their correctness must be obvious. Mathematical truths are commonly accepted as a model for scientific truths. Mathematical theorems seem to be convincing in a degree that is not comparable to the theorems of experimental sciences. It is so because they are accepted without appealing to experiments and only on the basis of the existence of a proof. Usage of computers in mathematical proofs changes the situation, because it introduces an experiment to mathematics – namely a computer experiment. Some philosophers claim that believing in computers can be compared to believing in some authority. T.Thymoczko in [4] compares the usage of computers in mathematical theorem proving on the Earth to the appeal by Martians of some hypothetical mathematical genius called Simon, who justifies theorems (or lemmas) on the basis of his personal verification moral. Each statement about which Simon said, that it is true, is accepted as true by other mathematicians and labeled: "Simon said-theorems". Thymoczko claims that such a practice on Mars can be treated as a good analogy for acceptance of computer proven theorems on the Earth.

This analogy is very often criticized by other authors cf. for example [5] and [6]. To compare conviction about correctness of results obtained by appealing to computers with believing in the authority of Simon is wrong. Nobody claims that computer proven theorems are true because computers are a kind of authority. We can check results in many ways:

– one can check correctness of the program,
– one can execute program on several different machines,
– one can write different programs resolving a considered problem and compare results.

Checking out machines is a domain of engineering, an engineer can find out if computer does what it is expected to do. Moreover, we can execute the same program several times on different machines and the same results can ascertain us to the correctness of the result.

If we are sure that the computer works correctly we can examine the program by checking correctness of the algorithm and its implementation. Checking the correctness of algorithms is not a difficult matter, and was done a long time before appearance of computers. Correctness of implementation of the algorithm can be checked directly (by looking up programs written in high-level programming languages) or by writing different implementation of the algorithms and comparing results. In the case of four-color theorem such a verification was made by Appel and Haken by the use of different programs on different computers.

It is true that many programs have gaps which remain unnoticed for months or even for years, but this is also the case in the classical mathematical proofs. "Proof" of four-color hypothesis published in 1879 had been considered as correct for almost 10 years, and nobody noticed a gap in the reasoning. So having gaps is a feature not only of computer proofs, but mathematical proofs as such.

We can also write a different program, based on different algorithm, which resolves the same problem and then compare results. In the case of four-color theorem there are two different computer proofs based on different algorithms, which undoubtedly make the proof of Appel and Haken more convincing.[4]

Consider why we so easily accept traditional proofs made by man, proofs very often checked only by one person, and we do not accept computer proofs checked by other computers. Notice that a human being becomes tired quickly and then often makes mistakes. So, if a proof of a theorem was checked by one person only is it more convincing than a proof done and checked by a computer? We do not think so. In computer proofs, on the contrary to traditional ones, probability of a mistake is smaller. Computers never get tired and almost never make mistakes if we check correctness of algorithm and implementation. Moreover, checking of the correctness of a proof is valid always only with some probability and it does not matter if this checking was made by a machine or a man.

4. The philosophical consequences of admittance of computer-assisted proofs as acceptable in mathematics

The most important consequence of admittance of computer proven theorems is the necessity of changing the notion of a proof. It could be done by:

- introducing a new method of proving theorems – a computer experiment,
- admitting in proofs some fragments proven by computer.

Theorems proven by computer introduce to mathematics a new kind of setting up mathematical truths. It is only a matter of terminology if we call it an experiment or a method of proving. Obviously, in both cases to accept these methods, we must believe in correctness of computations made by computers.

Common acceptance of artificial proofs and computer-assisted proofs, make us modify or abandon some philosophical claims about mathematics, in particular the following claims:

- All mathematical theorems are known a priori.
- Mathematics, as opposed to natural science, has no empirical content.

[4] Different proofs of the four-color theorem were published by Allaire in 1993 and by Robertson, Sanders, Seymour and Thomas in 1997.

– Mathematics relies only on proofs, whereas natural sciences make use of experiments.
– Mathematical theorems are certain to a degree that no theorem of natural science can match.

A widely shared philosophical assumption is that there is a gulf between mathematics and natural sciences. Philosophers claim that mathematical knowledge is *a priori*, innate, formal, analytic and certain, whereas natural science knowledge is *a posteriori*, learned, empirical, synthetic and dubietable. Does the acceptance of computer proven theorems in mathematics change this convictions?

If we define traditionally *a priori* truths as those which can be known independently of any experiment contrary to *a posteriori* truths which can be known only on the basis of a particular experiment, then computer proven theorems can be considered as *a posteriori* truths.

But the term a priori as applied to truth can have at least two meanings:

1. A truth that possesses universal and necessary validity; truth that holds (is true) in all possible worlds.
2. A truth whose validity can be established without recourse to sense experience of physical world (truth which can be known only by "pure reasoning").

These two definitions are often treated as equivalent and are used interchangeably. But in the case of computer proven theorems a differentiation between them is very important, because if we accept the second definition then computer proven theorems are examples of empirical (*a posteriori*) mathematical truths.

If we define *a priori* truths as those, which can be known only by "pure reasoning", then computer assisted or automatic proofs do not establish a priori truths, because by pure reasoning we cannot obtain a part or a whole of computer proof. Our knowledge about four-color theorem is based on the results of experience, so on empirical premises. Moreover, it is not so probable that somebody will ever know this theorem by pure reasoning, because all proofs we have till today are based on computations carried out by computers. Thus, the four-color theorem is an example of mathematical truth which is *a posteriori* and probably will never be a priori. In terms of definition 2 computer proven theorems are part of mathematics which can be known only *a posteriori*, moreover, they introduce an empirical element to mathematics.

Let us now consider definition 2 in the case of mathematics, by comparing computer proven theorems with other mathematical truths.

There are many theorems in mathematics, whose proofs require checking cases. The scheme of such proofs is simple; in the course of justification we come to some place, which is so long or complicated that it is impossible to reach it only by pure reasoning. Then we must appeal to some additional resources: a pen and a piece of paper, a calculator or even a computer. Thus, such theorems can be classified according to what kind of resources had been used in their justification and borders between such classes will not be clear. Very often in order to make proof "in head" (by pure reasoning) we have to use a pen and a piece of paper to reduce the problem to simpler cases. Often, even if it is possible to make some justification in head, it is done by computer because it is a more comfortable and faster method. We can classify the four-color theorem as a truth which cannot be justified without appealing to computer. But this status can change! It is unreasonable to classify theorems as *a priori* or *a posteriori* on the basis of their present place in such a hierarchy.

Some philosophers (see [5] and [6]) claim that if we accept definition 2, we should recognize that the problem of the status of mathematical knowledge is, in fact, not new: it refers to the time when first mathematician made some calculations with a stick on sand. Computer calculations are treated by some philosophers only as a more complicated and sophisticated form of calculations than those made with a pen or with a stick.

The other example to support the claim that definition 2 is not sufficient in mathematics, is the case of verification if a given natural number is a prime is commonly known, that for small numbers such checking can be done in head, but for very big numbers the only way to do this is to appeal to computer. So how can we agree that some truths (as being the prime) are in some cases *a priori* and in others *a posteriori*?

Consequently, it seems to be obvious that definition 2 is not proper for classification of mathematical truths.

In terms of definition 1, *a priori* truths possess universal and necessary validity, which is independent of their present proofs. It does no matter if such proof is made by a man by pure reasoning or by a computer.

There are some opponents to rejected definition 2, because it is used in almost all sciences. They propose some modifications of this definition in order to admit that some truths *a priori* can be known by appealing to some kinds of experiments. But such attitude leads to difficulties with the definition of posteriori truths. A new definition of *a posteriori* truth was proposed in [5]:

> *" A posteriori truth is one whose truth is contingent upon the nature of the universe to which it applies and cannot, in principle, be known without carrying out at least some experiments."*

In this case the difference between *a priori* and *a posteriori* truths does not depend on appealing to experiments but on the nature of these truths. Moreover, the nature of experiments in the case of *a priori* and *a posteriori* truths is completely different. Experiments necessary for admitting *a posteriori* truths consist of measurement of some physical properties as temperature, pressure or velocity, while an experiment in the case of *a priori* truth includes manipulations on numbers and logical symbols.

In terms of such distinction, the four-color theorem and other computer proven theorems cannot be considered as *a posteriori* truths, because a computer experiment was used in their justification and the nature of such experiment is different from measurement of physical values.

Thus it should be clear that decision if a given truth is *a priori* or *a posteriori* is not unequivocal and depends on the definition of the term *a priori*.

At the end, let us consider if the introduction of computer proven theorems to mathematics is the beginning of revolutionary changes in this science? Will mathematics be another empirical science? Will the usage of computers in proving theorems create a new paradigm of doing mathematics?

Mathematics is still the deductive science in spite of frequent usage of computers. Computations carried out by computers are programmed by a man and the author watches over their correctness. Machine only performs some long or complicated computations. But for many mathematicians and philosophers such usage of computers is unacceptable, because, as was indicated above, a problem of reliability of such computations remains unsolved.

Notice that today many mathematical theorems have long and complicated proofs. Thus using computers in justification is only the result of the evolution of mathematics and problems considered by mathematicians. Introducing computers to mathematics is only the next step in evolution of this science, similar to introducing abacus or calculators.

But the fact is, that with the aid of computers many new facts in mathematics have been proven. Thus, comparing computers to abacus seems to be simplification of the problem of the status of computers in mathematics.

5. Conclusions

Summing up, we can say that incorporating computers to mathematics, especially in proving theorems, reveals many philosophical problems, such as the status of mathematical truths (distinction between *a priori* and *a posteriori*), the status of mathematics as a standard for formal science and finally the problem of acceptable methods of "doing" mathematics. Most of these problems remain unsolved because their solutions very often depend on interpretation of such terms as *a priori* or *a posteriori*.

We claim that computers are and will remain the only tools in human hands till the time when we are able to teach computers to think independently. To this moment mathematics remains the deductive science.

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