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Annales UMCS Informatica AI 2 (2004) 243-250

Annales UMCS
Informatica
Lublin-Polonia
Sectio AI

http://www.annales.umcs.lublin.pl/

Efficiency of 2–order iterative methods

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Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. New the iterative 2-order method for this problem is presented. The article includes the numerical results for the method.

1. Problem

Let $F : D \subset R^n \to R^n$ be a nonlinear operator, D is an open set. The point $x_0 \in D$ is given.

The problem of solving a system of nonlinear equations consists in finding a solution $x^* \in D$ such, that

$$F(x^*) = 0 . \tag{1}$$

Such nonsingular problems were considered in [1,2].

In this paper we give some results for singular problems.

Definition 1. Operator F is regularly singular at the point x^*, if exists c>0 such, that

$$\underset{x \in D}{\forall} \; \| F(x) \| \geq c \, \| x - x^* \|^2 . \tag{2}$$

Note 1. If there exists i, $1 \leq i \leq n$ such, that matrix $F_i^{'}(x^*)$ is nonsingular, then the operator F is regularly singular at the point x*.

Definition 2. The sequence $\{x_k\}$ is locally M-superquadratically convergent, if:

$$\lim_{k \to \infty} \frac{\| F(x_{k+1}) \|}{\| x_{k+1} - x_k \|^2} = 0 . \tag{3}$$

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Note 2. The condition (3) is equivalent to the following condition

$$\| F(x_{k+1}) \| \leq c_k \cdot \| x_{k+1} - x_k \|^2 \quad \text{with} \quad \lim_{k\to\infty} c_k = 0 . \tag{4}$$

For the singular system of equations the Newton method is at most Q-linearly convergent. For the regularly singular system of equations condition (3) guarantees Q-superlinear convergence of the method [3].

2. New methods

Definition 3. Method (I)

Given $x_k \in R^n$, we solve the equation

$$F(x_k) + F'(x_k)s_k + \frac{1}{2} s_k^T B_k s_k = 0 \tag{5}$$

and compute

$$x_{k+1} = x_k + s_k , \ \text{k=0,1,…} .$$

The operator $F''(x_k)$ will be aproximated by the operator $B_k = [B_k^1, B_k^2, ..., B_k^n], B_k^i = (B_k^i)^T , \ B_k^i \in R^{n\times n}$ for i=1,2,...,n.

Results: [3] If the operator B_k satisfies the secant equation

$$B_{k+1} s_k = F'(x_{k+1}) - F'(x_k) \tag{6}$$

and

$$\lim_{k\to\infty} \| B_{k+1} - B_k \| = 0 \tag{7}$$

then

1. If the system of equations is nonsingular, then method I is locally M-superquadratically convergent.
2. For the regularly singular system of equations, method I is superlinearly convergent to x*:

$$\| x_{k+1} - x^* \| \leq \alpha_k \| x_k - x^* \|, \quad \lim_{k\to\infty} \alpha_k = 0 . \tag{8}$$

The system of equations (5) is difficult to solve, because there are n quadratic equations with n variables.

We propose the Newton method for solving this system.

Algorithm: Given: $x_0 \in D$, eps>0,

$$B_0^i \in R^{n\times n}, \quad (B_0^i)^T = B_0^i , \quad B_0^i \cong F_i''(x_0) \quad \text{for i=1,2,…,n.}$$

maxiter – integer, maxiter ≥ 0.

Denote:

$$G(s_t) = F(x_k) + F'(x_k)s_t + \frac{1}{2}(s_t, B_k s_t) \tag{9}$$

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1. $s_k^0 := 0$; t:=0;
2. while $\left(\left\|G(s_k^t)\right\| > eps\right) \wedge \left(t \le \max iter\right)$

do
begin
a) solve (for Δs_k):

$$G'(s_k^t)\Delta s_k = -G(s_k^t)$$

$$s_k^{t+1} = s_k^t + \Delta s_k$$

b) t:=t+1;
end

If maxiter=0, then the above algorithm is equivalent to the Newton method. The operator B_k is defined as

$$B_{k+1}^i = B_k^i + \frac{E_k^i + (E_k^i)^T}{2} \tag{10}$$

where $E_k^i = \frac{r_k^i s_k^T}{s_k^T s_k}$, $r_k^i = \nabla F_i(x_{k+1}) - \nabla F_i(x_k) - B_k^i s_k^T$, for i=1,2,...,n.

3. Regularly singular system of equations

Consider the operator $F : R^3 \to R^3$:

$$F(x_1) = (x_1 - 1)^2 + x_2^2 + x_3^2 - 1$$

$$F(x_2) = (x_1 + 1)^2 + x_2^2 + x_3^2 - 1$$

$$F(x_3) = x_1 + x_2 + x_3$$

Calculate norm of the operator F:

$$\left\|F(x)\right\|^2 = \left(F(x_1)\right)^2 + \left(F(x_2)\right)^2 + \left(F(x_3)\right)^2 =$$

$$= \left((x_1 - 1)^2 + x_2^2 + x_3^2 - 1\right)^2 + \left((x_1 + 1)^2 + x_2^2 + x_3^2 - 1\right)^2 + \left(x_1 + x_2 + x_3\right)^2 =$$

$$= \left(x_1^2 - 2x_1 + x_2^2 + x_3^2\right)^2 + \left(x_1^2 + 2x_1 + x_2^2 + x_3^2\right)^2 + \left(x_1 + x_2 + x_3\right)^2 =$$

$$= \left(x_1^2 + x_2^2 + x_3^2\right)^2 - 4x_1\left(x_1^2 + x_2^2 + x_3^2\right)^2 + \left(x_1^2 + x_2^2 + x_3^2\right)^2 + 4x_1\left(x_1^2 + x_2^2 + x_3^2\right)^2 +$$

$$+ \left(x_1 + x_2 + x_3\right)^2 = 2\left(x_1^2 + x_2^2 + x_3^2\right)^2 + \left(x_1 + x_2 + x_3\right)^2 \ge 2\left(x_1^2 + x_2^2 + x_3^2\right)^2 = 2\left\|x\right\|^2 =$$

$$= 2 \left\| x - x^* \right\|^2, \qquad where \;\; x^* = [0,0,0]^T.$$

Hence

$$\left\|F(x)\right\| \ge \sqrt{2}\left\|x - x^*\right\|^2,$$

so the operator F is regularly singular at the point x*.

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This problem can not be solved by the Newton method, because the matrix $F'(x^*)$ is singular.

4. Experimental results

We consider singular problems and nonsingular system of equations.

Problem1.

$$F(x_1) = (x_1 - 1)^2 + x_2^2 + x_3^2 - 1$$
$$F(x_2) = (x_1 + 1)^2 + x_2^2 + x_3^2 - 1$$
$$F(x_3) = x_1 + x_2 + x_3$$

Problem 2.

$$F(x_1) = x_1^2 + x_2^2 + x_3^2 - 1$$
$$F(x_2) = 2x_1^2 + x_2^2 - 4x_3$$
$$F(x_3) = 3x_1^2 - 4x_2 + x_3^2$$

Problem 3.

$$F(x_1) = x_1^2 + 8x_2 - 16$$
$$F(x_2) = x_1 - e^{x_2}$$

Problem 4. Rosenbrock Function [4]

$$F(x_1) = 1 - x_1$$
$$F(x_2) = 10(x_2 - x_1)^2$$

Problem 5. Powell Singular Function

$$F(x_1) = x_1 + 10x_2$$
$$F(x_2) = \sqrt{5}(x_3 - x_4)$$
$$F(x_3) = (x_2 - 2x_3)^2$$
$$F(x_4) = \sqrt{10}(x_1 - x_2)^2$$

Problem 6. Wood Function

$$F(x_1) = -(200x_1(x_2 - x_1^2)) - (1 - x_1)$$
$$F(x_2) = 200(x_2 - x_1^2) - 20.2(x_2 - 1) + 19.8(x_4 - 1)$$
$$F(x_3) = -(0.25x_3(x_4 - x_3^2)) - (1 - x_3)$$
$$F(x_4) = 180(x_4 - x_3^2) - 20.2(x_4 - 1) + 19.8(x_2 - 1)$$

Problem 7. Brown Almost-Linear Function

$$F(x_i) = x_i + \sum_{j=1}^{n} x_j, \qquad i = 1,2,...,(n-1)$$

$$F(x_n) = \prod_{j=1}^{n} (x_j - 1)$$

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Problem 8. Artificial Test Problem

$$F(x_1) = e^{(x_1)^2} + x_2^2 - 3$$
$$F(x_2) = x_1 + x_2 + \sin(3x_1 + x_2)$$

Problem 9. Freudenstein And Roth

$$F(x_1) = -13 + x_1 + (x_2^3 - 5x_2^2 + 2x_2)$$
$$F(x_2) = -29 + x_1 + (x_2^3 + x_2^2 + 14x_2)$$

Table 1

Problem number	Start point X_0	Maxiter	Number of iterations k	Number of solving system equations	Newton method
1	[1,2,1]	0	14	14	15
		1	7	14	
		2	5	15	
		3	4	15	
1	[1,1,1]	0	14	14	*
1	[0.5,0.5,1]	0	13	13	14
		1	7	14	
		2	5	14	
		3	4	14	
1	[0,1,0]	0	11	11	12
		1	6	12	
		2	4	12	
1	[2,3,1]	0	14	14	15
		1	7	14	
		3	4	15	
		4	3	15	
1	[0.001,0.002,0.001]	0	4	4	5
		1	2	4	
2	[0.5,0.5,0.5]	0	4	4	14
		1	2	4	
		2	2	5	
		3	1	4	
2	[1,1,0]	0	4	4	12
		1	2	4	
		2	2	5	
		3	1	4	
2	[0.1,0.1,0.1]	0	7	7	*
		1	4	8	
		2	3	8	
		3	2	8	

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Problem number	Start point X_0	Maxiter	Number of iterations k	Number of solving system equations	Newton method
2	[0.01,0,0.05]	0	10	10	28
		1	5	10	
		2	4	11	
		3	3	11	
3	[1,2]	0	5	5	6
		1	4	8	
3	[0.1,0.1]	0	5	5	6
		1	4	8	
3	[1,0]	0	5	5	6
		1	4	8	
		2	3	9	
3	[1,1]	0	4	4	5
		1	3	6	
		2	2	5	
4	[0.1,0.3]	0	10	10	11
		1	5	10	
		2	4	11	
4	[0.001,0.003]	0	3	3	3
		1	2	4	
		3	1	3	
5	[1,0.9,1,0.9]	0	11	11	12
		1	6	12	
		2	4	12	
		3	3	12	
5	[1,0,1,0]	0	12	12	13
		1	6	12	
		2	4	12	
		3	3	12	
6	[0,0,0,0]	0	14	14	15
6	[0.1,0.1,0.1,0.1]	0	10	10	11
6	[1,1,0,0]	0	12	12	13
		1	12	24	
		3	6	21	
		4	4	16	

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Problem number	Start point X_0	Maxiter	Number of iterations k	Number of solving system equations	Newton method
7	[0,0,0]	0	6	6	*
		1	6	12	
		2	4	11	
7	[0.5,0.5,0.5]	0	8	8	*
		1	7	14	
		2	6	17	
		3	5	17	
8	[1,1]	0	7	7	9
8	[1,-1]	0	∞		∞
		1	5	10	
		2	4	11	
8	[0.1,0.1]	0	∞		∞
		1	∞		
		2	∞		
		3	21	81	
		4	4	15	
		5	4	15	
9	[0,0]	0	4	4	5
		1	3	6	
		2	3	8	
9	[1,2]	0	4	4	5
		1	3	6	
		2	3	8	
9	[20,1]	0	3	3	4
		1	2	4	
		2	2	5	

* – singularity

∞ – method is not convergent

Number of iterations k is equal to the number of calculation operator *F* and *F'*.

5. Results

The proposed method for regularly singular system of equations is locally superlinearly convergent while the Newton method diverges or is at most linearly convergent. In our method the number of calculation operator F and F' is generally less than in the Newton method even for nonsingular problems.

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Acknowledgments

We would like to thank Professor Marcin Paprzycki from Oklahoma State University for his helpful remarks and examples of the system of nonlinear equations.

References

[1] Grzegórski S.M., *Rescaled multilevel least-change almost secant methods*, Matematyka Stosowana, XXXIII, (1991).
[2] Grzegórski S.M., *Orthogonal projections on convex sets for Newton-like methods*, SIAM J.Numer. Anal., 22 (1985) 1208.
[3] Grzegórski S.M., Łukasik E., *Theory of convergence for 2-rank iterative methods*, prepared for publication.
[4] Dent D., Paprzycki M., Kucaba-Pietal A., *Testing convergence of nonlinear system solvers*, Proceeding of the First Southern Symposium on Computing, The University of Southern Mississippi, (1998).

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