



Evaluating linear recursive filters on clusters of workstations

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Abstract

The aim of this paper is to show that the recently developed high performance algorithm for solving linear recurrence systems with constant coefficients together with the new BLAS-based algorithm for narrow-banded triangular Toeplitz matrix-vector multiplication allow to evaluate linear recursive filters efficiently, even on clusters of workstations. The results of experiments performed on a cluster of twelve Linux workstations are also presented. The performance of the algorithm is comparable with the performance of two processors of Cray SV-1 for such kind of recursive problems.

1. Introduction

Let us consider the following problem of evaluating linear recursive filters which is very popular in signal processing [1]. For a given sequence of real numbers x_1, x_2, \dots, x_n called input signals, we have to evaluate an output sequence y_1, y_2, \dots, y_n , satisfying

$$y_k = \sum_{j=1}^m b_j y_{k-j} + \sum_{j=0}^m a_j x_{k-j}, \quad (1)$$

where $x_k = 0$, $z_k = 0$ for $k \leq 0$ and coefficients a_j , b_j are calculated using z -transforms [1]. Unfortunately, the problem has “recursive” nature and simple routines based on (1) do not fully utilize the underlying hardware, i.e. memory hierarchies and multiple processors, and they achieve poor performance. On the other hand, our problem is a typical example of how the speed of the slowest parts of programs influences the overall performance (so called *Amdahl's law* [2]). Thus, it is clear that efficient high performance algorithms for solving our problem should be designed.

It is well known that reducing costs of memory access is essential for achieving good performance of numerical software [3]. Usually, codes based on Level 2 and 3 BLAS (Basic Linear Algebra Subprograms [2]) routines achieve

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$$\mathbf{x}_j = (x_{(j-1)s+1}, \dots, x_{js})^T \text{ and } \mathbf{f}_j = (f_{(j-1)s+1}, \dots, f_{js})^T \in \mathbb{R}^s$$

and matrices

$$L = \begin{pmatrix} a_0 & & & & \\ \vdots & \ddots & & & \\ a_m & & \ddots & & \\ & & & \ddots & \\ & & & & a_m & \cdots & a_0 \end{pmatrix}, \quad U = \begin{pmatrix} & a_m & \cdots & a_1 \\ & & \ddots & \vdots \\ & & & a_m \\ 0 & & & \end{pmatrix} \in \mathbb{R}^{s \times s}.$$

Then all vectors \mathbf{f}_j satisfy

$$\begin{cases} \mathbf{f}_1 = L\mathbf{x}_1 \\ \mathbf{f}_j = U\mathbf{x}_{j-1} + L\mathbf{x}_j \text{ for } j = 2, \dots, r. \end{cases} \quad (5)$$

Note that we can use (5) to compute f_1, \dots, f_{rs} and (2) to find f_{rs+1}, \dots, f_n . When we define matrices

$$F = (\mathbf{f}_1, \dots, \mathbf{f}_r), \quad X = (\mathbf{x}_1, \dots, \mathbf{x}_r) \in \mathbb{R}^{s \times r}$$

and

$$G = \begin{pmatrix} a_m & a_{m-1} & \cdots & a_1 \\ 0 & a_m & \cdots & a_2 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_m \end{pmatrix} \in \mathbb{R}^{m \times m},$$

we can find vectors \mathbf{f}_j as follows. First we perform the operation $F \leftarrow LX$ and next we update first m entries of each vector $\mathbf{f}_2, \dots, \mathbf{f}_r$ using $GX_{s-m+1:s, j}$, $j=1, \dots, r-1$, respectively. Thus the algorithm comprises the following two steps:

Step A1:

$$F_{k,*} \leftarrow (a_{\min\{k-1, m\}}, \dots, a_1, a_0) \cdot X_{\max\{1, m-k\}; k, *} \text{ for } k = 1, \dots, r. \quad (6)$$

Step A2:

$$F_{1:m, 2:r} \leftarrow F_{1:m, 2:r} + GX_{s-m+1:s, 1:r-1}. \quad (7)$$

Note that **Step A1** can be implemented as a sequence of calls to the Level 2 BLAS operation `_GEMV`, i.e. matrix-vector multiplication, while **Step A2** as the Level 3 BLAS operation `_TRMM`, i.e. triangular matrix - general matrix multiplication. The second stage of the algorithm for finding (1) can be done using the *divide and conquer* algorithm for solving (3), which is based on Level 2 and 3 BLAS routines `_GEMV`, `_TRMV` and `_GEMM` [5].

3. Implementation

The algorithm can be easily implemented on distributed-memory parallel computers and clusters of workstations using MPI [8]. Generally speaking, each processor is responsible for computing a block of output signals y_i . We assume

that each processor holds (or calculates) coefficients a_0, \dots, a_m and b_1, \dots, b_m . Also, each processor holds r/p columns of the matrix X , where p is the number of available processors. The first stage of the algorithm requires communication, namely each processor (except for the last one) sends m last entries of its last column of the block of X to its “left” neighbour. The details of the second stage of the algorithm can be found in [7].

It is obvious that the performance of the algorithm depends on the choice of the parameters s and r . The second stage is more complicated thus it is clear that the parameters should be chosen to minimize the cost of that stage. To predict the behaviour of the algorithm for solving (3), we have observed [7] that its complexity can be expressed in terms of Bulk Synchronous Parallel Architecture (BSP for short [9]). In this model, a parallel program consists of a number of supersteps. Each superstep comprises local computations, global data exchange and the barrier synchronization (see Figure 1).

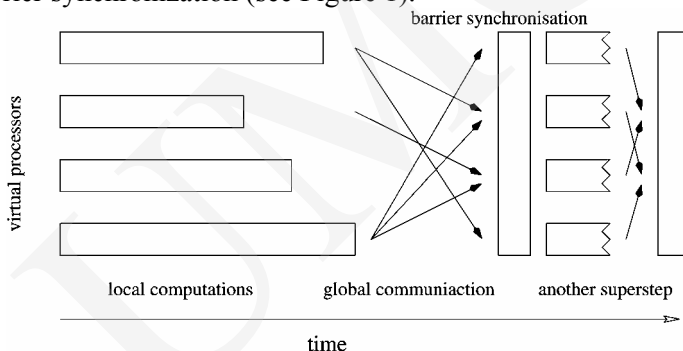


Fig. 1. BSP architecture

The BSP model is characterized by the following parameters: p , the number of available processors, g , the time (in flop time units) it takes to communicate (send or receive) a data element and l , the time (in flop time units) it takes all processors to synchronize. The complexity of a superstep is defined as

$$w_{\max} + gh_{\max} + l$$

where w_{\max} is the maximum number of flops performed, and h_{\max} is the number of messages sent or received by any one processor during this superstep.

To find the optimal value of s , we minimize the BSP cost of the “most time consuming” steps of the algorithm, namely

$$2m \left(s - \frac{m+1}{2} \right) \left(\frac{n}{sp} + 1 \right) + 3m^2 \frac{n}{s} + mg(p-1) + lp \tag{8}$$

and because the value of s should be integer, we get

$$s^* = \left\lceil \sqrt{\frac{n(3mp - m - 1)}{2p}} \right\rceil \tag{9}$$

Thus we can conclude that for the algorithm, the optimal choice of the value of the parameter s does not depend on the BSP parameters. So we can expect that the choice (9) will be good for a wide variety of BSP architectures and the formula (9) can be included in the source code of the algorithm.

4. Results of experiments

The algorithm has been tested on a cluster of twelve 667 MHz Pentium III workstations running under Linux operating system for various problem sizes (n , m) and the number of processors (p). The results of experiments can be summarized as follows.

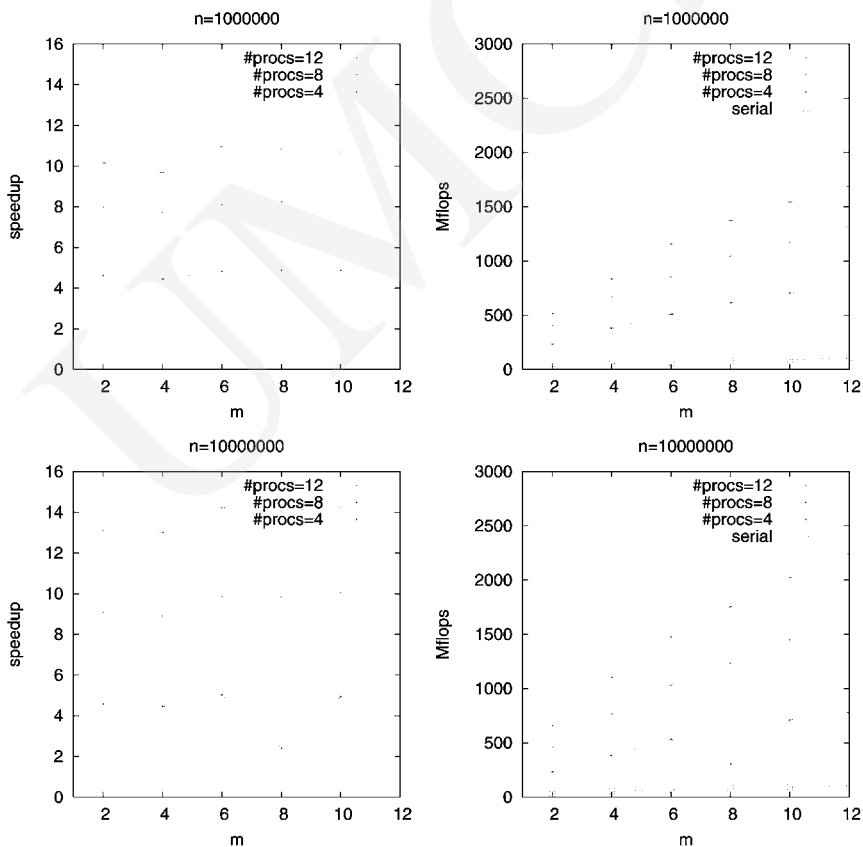


Fig. 2. The performance of the algorithm

1. The algorithm achieves the best performance for the values of $s=0.5s^*$. The performance of the algorithm depends strictly on the performance of the BLAS routines (which is not taken into account in BSP) and for such values of s we achieve “the perfect balance” between the BSP cost and the

efficiency of the BLAS routines. When $s=s^*$, then the matrices are too narrow.

2. The performance of the algorithm (in Mflops) grows when the problem sizes (n, m) and the number of processors grow. However the performance of simple serial algorithm based on (1) also grows when m grows. Thus we can observe that the speedup is almost constant when m grows. For $p=12$, the speedup of the algorithm is up to 15.
3. When $p=12$, then the performance of the algorithm (in Mflops) is up to 2200 Mflops, which is comparable with the performance of two processors of Cray SV-1 [6] for solving such a kind of recursive problems.

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