Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46

**Annales UMCS Informatica AI 2 (2004) 269-281**

**Annales UMCS Informatica Lublin-Polonia Sectio AI**

http://www.annales.umcs.lublin.pl/

# High-order extremal principles and *P*-factor penalty function method for solving irregular optimization problems

Ewa Szczepanik, Alexey Tret'yakov

*Department of Natural Sciences, University of Podlasie, 08-110 Siedlce, Poland*

**Abstract**

There is given a description of the solution set to p-regular equality – constrained optimization problems. Based on the apparatus of factor-operators P-order conditions for optimality are presented. The method for solving irregular optimization problems is proposed.

## 1. Introduction

In this paper we are concerned with a new class of methods for solving irregular optimization problems with equality constraints.

To analyse these methods the higher order optimality conditions are applied. We construct the new form of p-factor penalty function method, which permits us to reduce the constrained optimization problem to the series of unconstrained problems. These conditions are obtained in the frameworks of recent development of the p-regularity theory. Some of these results have been presented at the French-German-Polish Conference on Optimization, Cottbus, September 9-13, 2002.

Let us consider the problem of solving nonlinear optimization problem in the following form

$$\text{minimize } \phi(x) \qquad (1)$$
$$\text{subject to } F(x) = 0$$

where $\phi : R^n \to R^1$ is a sufficiently smooth function, $F : R^n \to R^m$ is a sufficiently smooth mapping, and the Jacobian matrix $F'(x^*)$ is singular in the solution $x^*$ to the problem (1).

The problem (1) is called regular in the solution $x^*$ if the Jacobian matrix has full rank, i.e.

$$rank\, F'(x^*) = m. \qquad (2)$$

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



In the other case, when the Jacobian matrix $F'\left(x^*\right)$ is singular, the problem (1) is called irregular (nonregular) at the point $x^*$ and $x^*$ is called singular (degenerate) solution to the problem.

The proof came as a result from the convergence properties of the methods for solving optimization problems, the regularity condition (2) is assumed to be held. When the regularity condition is violated many methods lose their high convergence rate or become inapplicable for finding singular solutions.

Moreover, in the nonregular case, the Euler-Lagrange optimality conditions

$$\lambda_0 \varphi'\left(x^*\right) + F'\left(x^*\right)^T \lambda^* = 0 \tag{3}$$

are trivially satisfied with any $\lambda_0 = 0$ such that $F'\left(x^*\right)^T \lambda^* = 0$ and do not give any constructive information about the solution to the optimization problem (1). In this case condition (3) describes the kernel of the operator $F'\left(x^*\right)$ only. Hence, equality (3) could not be applied to find nonregular solutions to the optimization problem (1).

The constructions of p-regularity introduced in [1,2] give new possibilities for description and investigation of nonregular solutions to the degenerated nonlinear optimization problems. For the p-regular problems, new necessary and sufficient optimality conditions were derived in [1-3].

In this paper, we present new methods for solving nonregular nonlinear optimization problems. The methods are constructed on the basis of the optimality conditions for the *p*-regular optimization problems.

We denote by $L\left(R^n, R^m\right)$ the space of all linear operators from $R^n$ to $R^m$. Further, $KerL = \left\{x \in R^n \mid Lx = 0\right\}$ denotes the null space of a given operator $L : R^n \to R^m$ and $ImL = \left\{y \in R^m \mid y = Lx \text{ for some } x \in R^n\right\}$ is an image space.

Also, $L^T : R^m \to R^n$ denotes the adjoint (or transpose) of $L$, $M^\perp = \left\{h \in R^n \mid \langle h, x\rangle = 0 \quad \forall x \in M\right\}$ denotes annihilator of the set $M$ and $\langle x, y\rangle = x_1 y_1 + x_2 y_3 + ... + x_n y_n$.

Let $p$ be a natural number and let $B : \underbrace{R^n \times R^n \times ... \times R^n}_{p} \to R^m$ be a symmetric $p$-multilinear mapping. The $p$-form associated with $B$ is the map $B[.]^p : R^n \to R^m$ defined by

$$B[x]^p = B\underbrace{(x, x, ..., x)}_{p}$$

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



for $x \in X$. Alternatively, we can simply view $B[.]^p$ as a homogeneous polynomial map $B: R^n \to R^m$ of degree p, i.e., $B(\alpha x) = \alpha^p B(x)$. The space of homogeneous polynomial $B: R^n \to R^m$ of degree $p$ will be denoted by $Q^p(R^n, R^m)$.

If the mapping $F: R^n \to R^m$ is differentiable, then its derivative at a point $x \in X$ will be denoted as $F'(x): R^n \to R^m$. If $F(x): R^n \to R^m$ belongs to the class $C^p$, then we take $F^{(p)}(x)$ to be the $p$-th order derivative of $F$ at the point $x$ (a symmetric multilinear map of $p$ copies of $R^n$ to $R^m$ and the associated p-form, also called the $p$-th - order, is

$$F^{(p)}(x)[h]^p = F^{(p)}(x)\underbrace{[h,...,h]}_{p}.$$

Furthermore, we will use the following key notation,

$$Ker^p F^{(p)}(x) = \left\{h \in R^n \mid F^{(p)}(x)[h]^p = 0\right\}$$

for the $p$-kernel of the mapping $F^{(p)}(x)$.

## 2. Elements of P-regularity theory. P-factor operators

It is well known, that Lyusternik theorem [4] provides a useful tool for constructive description of the tangent cone to the set $M(x^*) = \left\{x \in R^n : F(x) = F(x^*)\right\}$ for the regular mapping $F$. Let us recall some definitions and the Lyusternik theorem.

We say the mapping $F$ is regular at the point $x^*$ if

$$\operatorname{Im} F'(x^*) = R^m. \tag{4}$$

The mapping F is called nonregular (degenerate, irregular) if regularity condition (4) is violated.

**Definition 1.** *Let $M$ be a subset of the Euclidean space $R^n$. A vector $h \in R^n$ is said to be tangent to the set $M$ at the point $x^*$ if there exist an $\varepsilon > 0$ and a mapping $t \to r(t)$ of the interval [0, $\varepsilon$] into $R^n$ such, that*

$$x^* + th + r(t) \in M \qquad \forall t \in [0, \varepsilon],$$

$$\lim_{t \to 0} \frac{\|r(t)\|}{t} = 0.$$

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



*A set of vectors tangent to the set* $M\left(x^*\right)$ *at the point* $x^*$ *is a closed cone. This cone is usually called the tangent cone to the set* $M\left(x^*\right)$ *at the point* $x^*$ *and is denoted by* $T_1M\left(x^*\right)$.

**Theorem 1.** (Lyusternik theorem) *Let* $R^n$ *and* $R^m$ *be Euclidean spaces, U be a neighborhood of the point* $x^* \in R^n$, *and* $F: U \to R^m$, $F \in C^1\left(U, R^m\right)$. *Furthermore let's assume that F is regular at* $x^*$.

*Then the tangent cone to the set* $M\left(x^*\right) = \left\{x \in U \mid F\left(x\right) = F\left(x^*\right)\right\}$ *at the point* $x^*$ *coincides with the kernel of the operator* $F'\left(x^*\right)$:

$$T_1M\left(x^*\right) = KerF'\left(x^*\right). \tag{5}$$

We consider the case when regularity condition (4) does not hold, but the mapping *F* is *p*-regular. For this case a generalization of the Lyusternik theorem has been derived in [1-4]. First of all, let us remind the definition of *p*-regularity and construction of *p*-factor -operator.

We construct here *p*-factor -operator under an assumption that the space $R^m$ can be decomposed into direct sum

$$R^m = Y_1 \oplus ... \oplus Y_p, \tag{6}$$

where $Y_1 = \operatorname{Im} F'\left(x^*\right)$, $Y_i = lin\ Im\ P_{Zi}F^{(i)}\left(x^*\right)[.]^i$, $i = \overline{2, p-1}$, $Y_p = Z_p$, $Z_i = \left(Y_1 \oplus ... \oplus Y_{i-1}\right)^{\perp}$, $i = \overline{2, p-1}$. By $P_{Zi}$ we denote orthoprojector onto $Z_i$, along $\left(Y_1 \oplus ... \oplus Y_{i-1}\right)$ with respect to $R^m$.

We introduce the new mappings

$$g_i\left(x\right): R^n \to Y_i,$$
$$g_i\left(x\right) = P_{Yi}F\left(x\right) \quad \text{for } i = \overline{1, p}$$

where $P_{Yi}$: orthoprojection onto $Y_i$.

**Definition 2.** *Linear operator*

$$\Psi_p\left(h\right) \in L\left(R^n, Y_i \oplus ... \oplus Y_p\right), \quad h \in R^n$$

$$\Psi_p\left(h\right) = g_1'\left(x^*\right) + \frac{1}{2!}g_2''\left(x^*\right)h + ... + \frac{1}{p!}g_p^{(p)}\left(x^*\right)[h]^{p-1}$$

*is called p-factor -operator.*

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



**Definition 3.** *We say the mapping F is p-regular at the point* $x^*$ *along an element* $h \in R^n$, *if p is the lowest integer number such that*

$$\mathrm{Im}\Psi_p(h) = R^m.$$

**Definition 4.** *We say the mapping F is p -regular at the point* $x^*$, *if it is p-regular along any element h from the set.*

$$H_P(x^*) = \left\{ \bigcap_{r=1}^{p} Ker^r g_r^{(r)}(x^*) \right\} \setminus \{0\}.$$

**Definition 5.** *Function*

$$L_p(x,h,\lambda_0(h),y(h)) = \lambda_0(h)\phi(x) + \sum_{i=1}^{p} \left\langle y_i(h), g_i^{(i-1)}(x)[h]^{i-1} \right\rangle$$

*is called p-factor Lagrange's function, where* $y_i(h) \subset Y_i \;\; i = \overline{1,p}$, $\lambda_0(h) \in R$.

The following theorem [3,4] will be used in our analysis:

**Theorem 2.** (Necessary conditions for optimality in singular case)

*Let U be a neighborhood of the point* $x^*$ *and* $\phi \in C^2(U,R)$, $F \in C^{p+1}(U,R^m)$. *Suppose* $h \in H_p(x^*)$. *If* $x^*$ *is a locmin (1), then there exist* $\lambda_0(h) \in R$ *and multipliers* $y^*(h) = (y_1^*(h),...,y_p^*(h))$, *such that they do not all vanish, and*

$$L'_{px}(x^*,h,\lambda_0(h),y^*(h)) = \lambda_0(h)\phi'(x^*) + \sum_{i=1}^{p} \left( g_i^{(i-1)}(x^*)[h]^{i-1} \right)^T \quad y_i^* = 0.$$

*If, moreover,* $\mathrm{Im}\Psi_p(h) = Y_1 \oplus ... \oplus Y_p$, *then* $\lambda_0(h) \neq 0$.

### 3. Sufficient conditions for optimality in the singular case

Define the mappings

$$\overline{f}_i^{\,p}(x,h) = \left( g_1(x) + g_2'(x)h + ... + g_p^{p-1}[h]^{p-1} \right)_i \quad i = \overline{1,m}.$$

Based on the above constructions and theorem we introduce so-called p - factor -penalty function:

$$M_p(x,K) = \phi(x) + K\sum_{i=1}^{m} \left| \overline{f}_i^{\,p}(x,h) \right|^r, \;\; r = 1,2..., \quad K > 0. \tag{7}$$

For the sake of simplicity we consider the case $p = 2$. For $p = 2$, $p$-factor penalty function has the following form

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



$$M_2(x,K) = K\sum_{i=1}^{m}\left|\left(g_1(x)+g_2'(x)h\right)_i\right|^r$$

where $\left(g_1(x)+g_2'(x)h\right)_i = \left(F(x)+P^{\perp}F'(x)h\right)_i$, $P^{\perp}$ is orthoprojection onto $\left(ImF'\left(x^*\right)\right)^{\perp}$ and $h \in Ker\ F'\left(x^*\right) \cap Ker^2 P^{\perp}F''\left(x^*\right)$, $\|h\|=1$.

In this case 2-factor Lagrange`s function will be as follows

$$L_2(x,h,1,y) = \phi(x) + \left\langle y, F(x)+P^{\perp}F'(x)h\right\rangle. \tag{8}$$

From Theorem 2 it follows that at the solution point $x^*$ must be fulfilled

$$L_{2x}'\left(x^*,h,1,y^*\right) = \phi'\left(x^*\right) + \left(F'\left(x^*\right)+P^{\perp}F''\left(x^*\right)h\right)^T y^* = 0. \tag{9}$$

It means that for the problem

$$\min \phi(x) \tag{10}$$

$$\text{subject to} \quad F(x)+P^{\perp}F'(x)h = 0 \tag{11}$$

condition (9) under some additional assumption will be the *necessary condition* for optimality at the point $x^*$. It is obvious that the 2-factor penalty function $M_2(x,K)$ will be the ordinary penalty function for the problem (10).

**Theorem 3.** (Sufficient conditions for optimality in the singular case)

*Let U be a neighborhood of the point* $x^*$, $\phi \in C^2(U,R)$, $F \in C^3\left(U,R^m\right)$ *and F 2-regular at the point* $x^*$ *along element* $h \in H_2\left(x^*\right)$

*If there exist* $\alpha>0$ *and multipliers* $y^*(h)$ *such that:*

$$L_{2x}'\left(x^*,h,1,y^*\right) = 0 \tag{12}$$

$$L_{2xx}''\left(x^*,h,1,y^*\right)[z]^2 \geq \alpha\|z\|^2$$

*for all* $z \in Ker\left(F'\left(x^*\right)+P^{\perp}F''\left(x^*\right)h\right)$, *then point* $x^*$*-is a strict local minimizer to the problem (10).*

From the fact that constraint (11) is regular in classical sense, the proof followed from the same result for the minimization problems with regular constraints [5].

## 4. Convergence and rate of convergence for the 2-factor penalty function method

Consider 2- factor penalty function for problem (10)

$$M_2(x,K) = \phi(x) + K\sum_{i=1}^{m}\left|\overline{f}_i^{\,2}(x,h)\right|^r.$$

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



Let $x_2(K)$ be a local solution to the following minimization problem

$$\min M_2(x,K), \quad x \in U_\varepsilon(x^*) \tag{14}$$

where $U_\varepsilon(x^*)$ - $\varepsilon$-neighborhood of the point $x^*$ and $\varepsilon$ - sufficient small.

Denoted by

$$\Delta_2(K) = \phi(x^*) - M_2(x_2(K),K)$$

the accuracy of the solution $x_2(K)$.

From the above given theorems we can postulate that the following theorem will be held.

**Theorem 4.** (Convergence and rate of convergence for the 2-factor penalty function method in the singular case)

*Let $x^*\wedge$ be a locmin to (1) and the following conditions are fulfilled:*

*a)* $\phi \in C^2(R^n)$, $F \in C^3(R^n)$,

b) F- 2-regular at the point $x^*$ along the element $h \in H_2(x^*)$,

c) necessary and sufficient conditions for optimality are fulfilled at the point $x^*$

$$L'_{2x}(x^*,h,1,y^*) = 0,$$

$$L''_{2xx}(x^*,h,1,y^*)[z]^2 \geq \alpha \|z\|^2$$

$$\forall z \in Ker(F'(x^*) + P^{\perp}F''(x^*)h).$$

*Then for sufficiently large $K_0 > 0$ there exists solution (14)*

$$x_2(K) = loc\min_{x \in U_\varepsilon(x^*)} M_2(x,K) \tag{15}$$

*for all $K \geq K_0$ and*

$$\Delta_2(K) = 0, \quad r = 1 \tag{16}$$

$$0 \leq \Delta_2(K) \leq C\left(\frac{1}{K}\right)^{\frac{1}{r-1}} \quad r = 2,3,..., \tag{17}$$

*where C > 0 is constant.*

**Proof.** It is obvious that for the problem

$$\min \phi(x)$$

$$\text{subject to} \quad \overline{f}_i^{\,2}(x,h) = 0, \quad i = \overline{1,m} \tag{18}$$

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



the point $x^*$ -is a strict local *minimizer* (Theorem 3). And the constraints $\overline{f}_i^2(x,h)$, $i=\overline{1,m}$ are regular at the point $x^*$ (since the matrix $\left(F'(x^*)+P^{\perp}F''(x^*)h\right)$ is nonsingular).

Then, obviously, the feasible set

$$D=\left\{x\in R^n \mid \overline{f}_i^2(x,h)=0,\quad i=\overline{1,m}\right\}$$

is 1-majorizable [6] in the $\varepsilon$-neighborhood $U_\varepsilon(x^*)$, i.e.

$$\rho(x,D)\le C_0\sum_{i=1}^{m}\left|\overline{f}_i^2(x,h)\right|, \tag{19}$$

where $\rho(x,D)=\inf\limits_{y\in D}\|x-y\|$, $C_0>0$ – constant.

We will prove, that there exists solution $x_2(K)$ to the problem (14) inside $U_\varepsilon(x^*)$: $x_2(K)\in intU_\varepsilon(x^*)$. At the solution $x^*$ we have

$$M_2(x^*,K)<M_2(z,K) \tag{20}$$

for all points $z\in SU_\varepsilon(x^*)=\left\{z\in U_\varepsilon(x^*)\mid\|z-x^*\|=\varepsilon\right\}$, ε > 0 – small enough.

Let $\Delta(D)=\left\{x\in U_\varepsilon(x^*)\mid\rho(x,D)\le\Delta\right\}$ be a delta strip of feasible set, (Fig. 1).

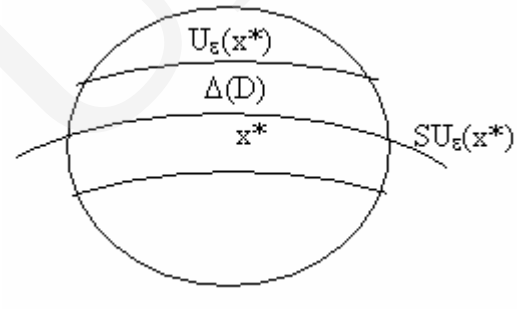



Fig. 1

The whole sphere may be represented as $SU_\varepsilon(x^*)=D_1\cup D_2$, where $D_1=\left\{\Delta(D)\cap SU_\varepsilon(x^*)\right\}$, $D_2=\left\{SU_\varepsilon(x^*)\setminus\Delta(D)\right\}$.

If we denote

$$0<\alpha=\min_{x\in D_2}\sum_{i=1}^{m}\left|\overline{f}_i^2(x,h)\right| \quad\text{and}\quad \beta=\max_{x\in D_2}\left(\phi(x^*)-\phi(x)\right)$$

then for the points $x\in SU_\varepsilon(x^*)\setminus\Delta(D)$ and sufficiently large $K$ we have $K\alpha>\beta$ and

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



$$\phi\left(x^*\right)+K\sum_{i=1}^{m}\left|\overline{f}_i^{\,2}\left(x^*,h\right)\right|<\phi(x)+\sum_{i=1}^{m}\left|\overline{f}_i^{\,2}\left(x,h\right)\right|,$$

$$K\sum_{i=1}^{m}\left|\overline{f}_i^{\,2}\left(x^*,h\right)\right|\geq K\alpha>\left(\phi\left(x^*\right)-\phi(x)\right)\quad\forall x\in D_2.$$

It means that

$$M_2\left(x^*,K\right)<M_2\left(z,K\right),\quad\forall z\in D_2. \tag{21}$$

Now we will show that inequality (20) holds for the points

$$z\in D_1=SU_\varepsilon\left(x^*\right)\cap\Delta(D).$$

From conditions (13) and (19) we have

$$\phi(x)+\sum_{i=1}^{m}\left|\overline{f}_i^{\,2}\left(x,h\right)\right|\geq\phi(z)+K\rho(z,D).$$

Let us denote by $y_z$ - ortoprojection $z$ onto $D$.

We have

$$\begin{aligned}\phi(x)&=\phi(y_z)+\left\langle\phi'(y_z),z-y_z\right\rangle+o\left(\left[z-y_z\right]\right)\geq\\&\geq\phi(y_z)-2\left|\left\langle\phi'(y_z),z-y_z\right\rangle\right|\geq\phi(y_z)-C\left\|z-y_z\right\|,\end{aligned} \tag{22}$$

where C > 0 is the independent constant, $\rho(z,D)=\left\|z-y_z\right\|$, and based on the relation $\phi\left(x^*\right)\leq\phi(y_z)$ we have

$$\phi(y_z)-C\left\|z-y_z\right\|\geq\phi\left(x^*\right)-C\left\|z-y_z\right\|.$$

From this

$$\begin{aligned}\phi(x)+K\sum_{i=1}^{m}\left|\overline{f}_i^{\,2}\left(z,h\right)\right|&\geq\phi\left(x^*\right)-C\rho(z,D)+K\frac{1}{C_0}\rho(z,D)=\\&=\phi\left(x^*\right)+\left(K\frac{1}{C_0}-C\right)\rho(z,D)\geq\phi\left(x^*\right)+K\sum_{i=1}^{m}\left|\overline{f}_i^{\,2}\left(x^*,h\right)\right|,\end{aligned}$$

where $\dfrac{K}{C_0}-C>0$ for $K\geq K_0=C_0*C$.

It means that

$$M_2\left(x^*,K\right)\leq M_2\left(z,K\right),\text{ for }z\in D_1. \tag{23}$$

Finally from (21) and (23) we have

$$M_2\left(x^*,K\right)\leq M_2\left(z,K\right),\quad\forall z\in SU_\varepsilon\left(x^*\right)$$

for

$$K\geq K_0>0.$$

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



It means that there exists a solution $x_2(K) \in intU_\varepsilon(x^*)$ for $K \geq K_0 > 0$, $r = 1$. Analogously we can prove the existence of $x_2(K)$ for $r = 2,3,\ldots$.

Now the existence of the solutions $x_2(K) \in U(x^*)$ allowed us to apply the Theorem 6.2.7 from [6] (pp.131-132) in order to estimate the convergence rate of our method. Under the above conditions the following equations will be satisfied

$$\Delta_2(K) = 0, \quad r = 1$$

and

$$\Delta_2(K) \leq C\left(\frac{1}{K}\right)^{\frac{1}{r-1}} \quad r = 2,3,\ldots$$

for all sufficiently large $K \geq K_0$.

The theorem is proved.

## 5. Example

Consider an optimization problem of the following form

$$\min x_2^2 + x_3 \tag{24}$$

with constraints

$$F(x) = \begin{cases} \frac{1}{2}\left(x_1^2 - x_2^2 + x_3^2\right) = 0 \\ \frac{1}{2}\left(x_1^2 - x_2^2 + x_3^2\right) + x_2 x_3 = 0 \end{cases}$$

where $x^* = 0$ is the solution to this problem, $x \in R^3$.

We would like to find out: is the *F(x)* regular at the point $x^* = 0$, i.e. are the gradients of the constraints linearly independent at $x^* = 0$?

Here

$$F'(x) = \begin{bmatrix} x_1 & -x_2 & x_3 \\ x_1 & -x_2 + x_3 & x_2 + x_3 \end{bmatrix}.$$

At the point $x^* = 0$ we obtain

$$F'(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case we see that the gradients of the constraints are linearly dependent and the mapping *F(x)* is not regular at the point $x^* = 0$.

Furthermore, we could not say anything about the convergence of classical penalty function and we could not guarantee the existence of *x(K)* for sufficiently large $K > 0$.

Now we modify our problem (24) in accordance with our theory

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



$$\begin{aligned} &\min x_2^2 + x_3 \\ &\text{subject to} \quad F(x) + P^{\perp} F'(x) h = 0, \end{aligned} \tag{25}$$

where $P^{\perp}$ - orthoprojection on to $\left(ImF'(0)\right)^{\perp}$ is equal to identity matrix and $h \in Ker^2 P^{\perp} F''(0)$, $\|h\| \neq 0$.

In our case we have

$$KerF'(0) \cap Ker^2 P^{\perp} F''(0) = Lin\left\{\begin{pmatrix}1\\1\\0\end{pmatrix} \quad \begin{pmatrix}1\\-1\\0\end{pmatrix}\right\},$$

and

$$F''(0) = \left(\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 1\\ 0 & 1 & 1\end{bmatrix}\right), \quad F'(0) = [0].$$

The Lagrange's function for the problem (25) has the following form

$$L(x, y) = \phi(x) + \langle y, F(x) + P^{\perp} F'(x) h \rangle.$$

Based on Theorem 2 we can derive Lagrange's multipliers. Since

$$F(x) + P^{\perp} F'(x) h = \begin{bmatrix} \frac{1}{2}\left(x_1^2 - x_2^2 + x_3^2\right) + x_1 - x_2 \\ \frac{1}{2}\left(x_1^2 - x_2^2 + x_3^2\right) + x_2 x_3 + x_1 - x_2 + x_3 \end{bmatrix}$$

for $h = (1,1,0)^T$, then

$$\begin{aligned} L_2(x, h, 1, y) = x_2^2 + x_3 + y_1\left(\frac{1}{2}\left(x_1^2 - x_2^2 + x_3^2\right) + x_1 - x_2\right) + \\ + y_2\left(\frac{1}{2}\left(x_1^2 - x_2^2 + x_3^2\right) + x_2 x_3 + x_1 - x_2 + x_3\right) \end{aligned}$$

and

$$L_2'(0, h, 1, y^*) = \begin{bmatrix}0\\0\\1\end{bmatrix} + y_1^* \cdot \begin{bmatrix}1\\-1\\0\end{bmatrix} + y_2^* \cdot \begin{bmatrix}1\\-1\\1\end{bmatrix} = 0.$$

We have obtained $y_1^* = 1$ and $y_2^* = -1$.

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



Taking into account the results of Theorem 3 we can verify positive definiteness of Hesse's matrix of the 2-factor Lagrange's function on the kernel $Ker\left(F'(0)+F''(0)h\right)$:

$$L_2\left(x,h,1,y^*\right)=x_2^2-x_2x_3$$

$$L_2''\left(0,h,1,y^*\right)=\begin{pmatrix}0&0&0\\0&2&-1\\0&-1&0\end{pmatrix}$$

$$Ker\left(F'(0)+F''(0)h\right)=lin\begin{pmatrix}1\\1\\0\end{pmatrix}$$

and obviously

$$\left\langle L_2''\left(0,h,1,y^*\right)z,z\right\rangle=2>0\,.$$

It means that $x^*=0$ is the local minimizer to the problem (25). Now we can construct the 2-factor penalty function for the problem (24)

$$M_2\left(x,K\right)=\phi\left(x\right)+K\,\|\,F\left(x\right)+P^{\perp}F'\left(x\right)h\,\|^r,\quad r=1,2,...$$

Consider the case $r = 1$.

$$M_2\left(x,K\right)=x_2^2+x_3+$$

$$+K\sqrt{\left|\frac{1}{2}\left(x_1^2-x_2^2+x_3^2\right)+x_1-x_2\right|^2+\left|\frac{1}{2}\left(x_1^2-x_2^2+x_3^2\right)+x_2x_3+x_1-x_2+x_3\right|^2}$$

From the equivalence of the norms we can consider the following form of $M_2\left(x,K\right)$:

$$M_2\left(x,K\right)=x_2^2+x_3+K\left(\left|\frac{1}{2}\left(x_1^2-x_2^2+x_3^2\right)+x_1-x_2\right|+\right.$$

$$\left.\left|\frac{1}{2}\left(x_1^2-x_2^2+x_3^2\right)+x_2x_3+x_1-x_2+x_3\right|\right).$$

Also we know that the mapping $F\left(x\right)+P^{\perp}F'\left(x\right)h$ is 1-regular at the point $x^* = 0$, so for solving problem (25) we can apply classical theorem about the convergence of the classical penalty function (Theorem 4).

Taking into account this theorem we obtain that:

There exists such $K_0>0$ that for $K\geq K_0$ will be fulfilled:

$$\Delta_2\left(K\right)\leq\left(\frac{C}{K}\right),\quad r=2\,,$$

where $C > 0$ arbitrary independent constant.

Pobrane z czasopisma Annales AI- Informatica **http://ai.annales.umcs.pl**
Data: 31/10/2026 03:26:46



Finally we can point out that from positive definiteness of Hessian matrix $L_2^{"}\left(0,h,1,y^*\right)$ for all $z \in KerF"(0)h$, $z \neq 0$, the existence of $x_2(K) \in U_\varepsilon\left(x^*\right)$ followed.

## 6. Conclusions

In this paper we have derived necessary and sufficient conditions for optimality to extremum problems in the presence of nonregular equality constraints. Our results are based on the *p*-regularity theory and apparatus of factor operators for constructive description of the structure of solution set in the singular case.

This allows us to apply these results to create new method for solving *p*-regular nonlinear optimization problems – *p*-factor penalty function method, to prove the convergence, and to estimate convergence rate of proposed method.

## References

[1] Tret'yakov A.A., *Necessary conditions for optimality of P-th order*, Control and Optimization, MSU, (1983), in Russian.
[2] Tret'yakov A.A., *Necessary and sufficient conditions for optimality of P-th order*, Computational Mathematics & Mathematical Physics, 24(1) (1984).
[3] Belash K.N., Tret'yakov A.A., *Methods for solving degenerate problems*, USSR Computational Mathematics & Mathematical Physics, 28(4) (1988).
[4] Izmailov A.F., Tret'yakov A.A., *Factor analysis of nonlinear mappings*, Moscow, Nauka, Fizmatlit Publishing Company, (1994) 336.
[5] Ioffe A.D., Tikhomirov V.M., *Theory of extremal problems*, North Holland, Amsterdam, The Netherlands, (1979).
[6] Karmanov V.G., *Mathematical programming*, Mir Publishers, Moscow, (1989).