Annales UMCS Informatica AI 3 (2005) 57-64

Annales UMCS
Informatica
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Sectio AI

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# The effective algorithm for solving the quadratic diofantic equation with three unknowns

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**Abstract**

The article is devoted to the problem of finding the triples of integer numbers generating equal sums of the squares in the linear combination. At first the theorem giving the conditions of existing two equal sums of the squares is presented. Based on it the algorithm for finding the triples of the integer numbers generating equal sums of the squares is built. The time complexity is calculated and the effectiveness for different size of the input parameters is verified.

## 1. Introduction

The aim of the article is finding the *3ths* of integer numbers generating equal sums of the squares in the linear combinations with positive rational coefficients. This problem is modeled by the following diofantic equation:

$$\frac{i_1^2}{m_1}+\frac{i_2^2}{m_2}+\frac{i_3^2}{m_3}=\frac{j_1^2}{m_1}+\frac{j_2^2}{m_2}+\frac{j_3^2}{m_3} \tag{1}$$

The effective numerical algorithm for solving equation (1) with the proper assumptions is presented and proved in the paper. Below we show as an example the 3 triples $(i_1,i_2,i_3)$ generating equal sum of the squares with parameters $m_1 = 1$, $m_2 = 3$, $m_3 = 5$:

$$\frac{143}{14}=\frac{2^2}{1}+\frac{4^2}{3}+\frac{1^2}{5}=\frac{1^2}{1}+\frac{4^2}{3}+\frac{4^2}{5}=\frac{3^2}{1}+\frac{1^2}{3}+\frac{1^2}{5} \tag{2}$$

Such a linear combination of the integer squares (1) exists in the eigenvalues multiplicity of the Laplace second order differential operator defined in the 3-dimensional rectangular prism. The knowledge of the multiplicity of these eigenvalues has important meaning in different dynamical properties of the

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dynamical system with the Laplace differential operator in its mathematical model.

## 2. Problem formulation

Let be given the following equation:

$$\frac{i_1^2}{m_1}+\frac{i_2^2}{m_2}+\frac{i_3^2}{m_3}=\frac{j_1^2}{m_1}+\frac{j_2^2}{m_2}+\frac{j_3^2}{m_3} \tag{3}$$

where:

$$m_1, m_2, m_3 \in Z_+ \tag{4}$$

are the given positive integer numbers.
The aim is: find all the non-symmetrical and non-trivial *2n*$^{\text{ths}}$:

$$(i_1,i_2,i_3,j_1,j_2,j_3)\in Z_+^6 \tag{5}$$

fulfilling the given equation (3), constrained by the given positive constant *N*:

$$\underset{k=1,2,3}{\forall} i_k, j_k \le N,\ N \in Z_+ \tag{6}$$

## 3. Solution

At first let us rewrite the given equation (3) in the form (7):

$$-\left(j_1^2-i_1^2\right)=\frac{m_1}{m_2}\left(j_2^2-i_2^2\right)+\frac{m_1}{m_3}\left(j_3^2-i_3^2\right) \tag{7}$$

The equation (3) in the form (7) appears to be suitable to introduce new variables, defined by the equation (8):

$$\begin{cases} s_k = j_k + i_k \\ r_k = j_k - i_k \end{cases},\ k=1,2,3 \tag{8}$$

Really, considering that (9):

$$j_k^2-i_k^2=\left(j_k-i_k\right)\left(j_k+i_k\right) \tag{9}$$

The equation (7) obtains the form (10):

$$-s_1r_1=\frac{m_1}{m_2}s_2r_2+\frac{m_1}{m_3}s_3r_3 \tag{10}$$

Summarising (10) to the common denominator we have (11):

$$r_1=-\frac{1}{s_1}\frac{m_1m_3s_2r_2+m_1m_2s_3r_3}{m_2m_3} \tag{11}$$

From (8) it can be calculated (12):

$$\begin{cases} i_k=\frac{1}{2}\left(s_k-r_k\right) \\ j_k=\frac{1}{2}\left(s_k+r_k\right) \end{cases},\ k=2,3 \tag{12}$$

Using (11) in (12) we have the formula (13) giving the values $(i_1,j_1)$ explicitly by the function of the parameters $s_k, r_k$, $k=2,3$ and the parameter $S_1$:

$$(i_1, j_1) = \frac{1}{2}\left( s_1 + \frac{1}{s_1}\frac{m_1 m_3 s_2 r_2 + m_1 m_2 s_3 r_3}{m_2 m_3}, s_1 - \frac{1}{s_1}\frac{m_1 m_3 s_2 r_2 + m_1 m_2 s_3 r_3}{m_2 m_3} \right) \tag{13}$$

Now let us calculate the constrains for the parameters, given in the formula (13). Considering that all the unknowns $(i_k,j_k)$, k = 1,2,3 must be positive integer values (5), and considering their upper limit (6) for the parameters $s_k, r_k$, $k=2,3$, we can determine the series of the set of inequalities (14):

$$\begin{cases} 1 \le \frac{1}{2}(s_k - r_k) \le N \\ 1 \le \frac{1}{2}(s_k + r_k) \le N \end{cases}, \ k = 2,3 \tag{14}$$

From (14) we have directly (15):

$$\begin{cases} r_k + 2 \le s_k \le 2N + r_k \\ -r_k + 2 \le s_k \le 2N - r_k \end{cases}, \ k = 2,3 \tag{15}$$

Each from the series of the sets of inequalities (15) produces a rectangle with the vertices (0,2),($N$-1,$N$+1),(0,2$N$),(–$N$+*1*,$N$+*1)* on a cartesian co-ordinate system $(r_k,s_k)$.

Now, let us find the constraints for the $s_1$ parameter. Applying the constraints (5) and (6) to the pair $(i_1,j_1)$ expressed by the formula (13) we have (16):

$$1 \le \frac{1}{2}\left( s_1 \pm \frac{1}{s_1}\frac{m_1 m_3 s_2 r_2 + m_1 m_2 s_3 r_3}{m_2 m_3} \right) \le N \tag{16}$$

which after a few calculations can be transformed into the form of four quadratic inequalities (17), (18):

$$m_2 m_3 s_1^2 - 2 m_2 m_3 s_1 \pm m_1 m_3 s_2 r_2 + m_1 m_2 s_3 r_3 \ge 0 \tag{17}$$

$$m_2 s_1^2 - 2N m_2 m_3 s_1 \pm m_1 m_3 s_2 r_2 + m_1 m_2 s_3 r_3 \le 0 \tag{18}$$

Following the aim of simplifying the notation let us introduce the variable $S$:

$$S(\vec{s}_2, \vec{r}_2) = m_1 m_3 s_2 r_2 + m_1 m_2 s_3 r_3 \tag{19}$$

where vectors $\vec{s}_2$ and $\vec{r}_2$ are given by:

$$\vec{s}_2 = (s_2, s_3), \vec{r}_2 = (r_2, r_3) \tag{20}$$

Now let us distinguish 2 cases:

### 3.1 Case 1: $S(\vec{s}_2, \vec{r}_2) \ge 0$

The determinant of the first two inequalities (17) using variable (19) has the form (21):

$$\Delta_1^{\pm} = 4 m_2^2 m_3^2 \pm 4 m_2 m_3 S(\vec{s}_2, \vec{r}_2) \tag{21}$$

From (21) there yields $\Delta_1^+ > 0$ (with the assumption in this case $S(\vec{s}_2,\vec{r}_2) \geq 0$), but the sign of $\Delta_1^-$ is not determined. If $\Delta_1^- \geq 0$ the solution of the inequalities (17) are the following two ranges:

$$P_1 = R \setminus \left(1 - \sqrt{1 - \frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}; 1 + \sqrt{1 - \frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}\right) \tag{22}$$

$$P_2 = R \setminus \left(1 - \sqrt{1 + \frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}; 1 + \sqrt{1 + \frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}\right) \tag{23}$$

Moreover, from (22), (23) there yields the following inclusion:

$$P_2 \subset P_1 \tag{24}$$

But the inequalities (17) and (18) must be fulfilled simultaneously, because they are equivalent to the constraint (16), so the solution of the inequalities (17) for $\Delta_1^- \geq 0$ is the logical product of the sets $P_1$,$P_2$. From (24) we can state that:

$$P_1 \cap P_2 = P_2 \tag{25}$$

so the solution in this case is the range (23). If $\Delta_1^- < 0$ the inequality (17) taken for + sign is fulfilled for all the real numbers, so also for all the integers, and the range (23) remains the solution of the inequalities (17). Summarising, in the case 1 the range (23) is the set of the solutions of the inequalities (17). Obviously we are only interested in integer points from the range $P_2$ (equation (8) and assumption (5)).

Now, let us pass to the inequalities (18). Its determinant is given by the formula (26):

$$\Delta_2^{\pm} = 4N^2 m_2^2 m_3^2 \pm 4 m_2 m_3 S(\vec{s}_2,\vec{r}_2) \tag{26}$$

From (18) there follows that if $\Delta_2^- < 0$ then the constraint (16) is not fulfilled. So if $\Delta_2^- \geq 0$ from the inequalities (18) we have two ranges (27), (28):

$$P_3 = \left[N - \sqrt{N^2 - \frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}; N + \sqrt{N^2 - \frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}\right] \tag{27}$$

$$P_4 = \left[N - \sqrt{N^2 + \frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}; N + \sqrt{N^2 + \frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}\right] \tag{28}$$

Now it can be noticed the following :

$$P_3 \subset P_4 \Rightarrow P_3 \cap P_4 = P_3 \tag{29}$$

and the common solution of the inequalities (18) is the range (27) with the condition $\Delta_2^- \geq 0$.

### 3.2. Case 2: $S(\vec{s}_2,\vec{r}_2)<0$

After similar calculations to those the point 0 we have the following range from the inequalities (17):

$$P_5 = R \setminus \left(1-\sqrt{1-\frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}};1+\sqrt{1-\frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}\right) \tag{30}$$

In this case from the two determinants given by (26) only $\Delta_2^+$ can be negative thus its sign should be checked, and the two inequalities (18) are equivalent to the range (31) and the condition (32) where $\Delta_2^+$ is given by the equality (26):

$$P_6 = \left[N-\sqrt{N^2+\frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}};N+\sqrt{N^2+\frac{S(\vec{s}_2,\vec{r}_2)}{m_2 m_3}}\right] \tag{31}$$

$$\Delta_2^+ \geq 0 \tag{32}$$

### 3.3. Simplifying the Parameter Ranges

The aim of this subchapter is to find the intersection of the ranges (23), (27) and (30), (31). Following this aim let us prove the following lemma:

#### 3.3.1. Lemma 0

The following inequalities hold:

$$1-\sqrt{1-a} < N-\sqrt{N^2+a}, \quad for\ a>0, N\geq 1 \tag{33}$$

$$1-\sqrt{1-a} > N-\sqrt{N^2+a}, \quad for\ a<0, N\geq 1 \tag{34}$$

#### 3.3.2. Proof

Let us take into account the following function:

$$f(a,N)=\left(1-\sqrt{1-a}\right)+\left(\sqrt{N^2+a}-N\right) \tag{35}$$

where $a\in R$ and $N$ is a positive integer number.

It can be seen that:

For $a>0$: $1-\sqrt{1-a}>0$ and $\sqrt{N^2+a}-N>0$, so $f(a,N)>0$ (36)

For $a<0$: $1-\sqrt{1-a}<0$ and $\sqrt{N^2+a}-N<0$, so $f(a,N)<0$ (37)

Moreover, there is only one zero of the function $f(a,N)$ for $a=0$. So from (35)-(37) we have (33) and (34).

Q.E.D.

Now, let us return to the main subject of this subchapter, i.e. simplifying the parameter ranges. The lemma 0 can be used after substitution (38):

$$a = \frac{S(\vec{s}_2, \vec{r}_2)}{m_2 m_3} \tag{38}$$

Using the inequalities (33), (34) we can finally reduce the ranges (23), (27) into one range (39):

$$P_7 = \left[ 1 + \sqrt{1 + \frac{S(\vec{s}_2, \vec{r}_2)}{m_2 m_3}}; N + \sqrt{N^2 - \frac{S(\vec{s}_2, \vec{r}_2)}{m_2 m_3}} \right] \ for\ S(\vec{s}_2, \vec{r}_2) \geq 0, \Delta_2^- \geq 0 \tag{39}$$

And the ranges (30), (31) into range (40):

$$P_8 = \left[ 1 + \sqrt{1 - \frac{S(\vec{s}_2, \vec{r}_2)}{m_2 m_3}}; N + \sqrt{N^2 + \frac{S(\vec{s}_2, \vec{r}_2)}{m_2 m_3}} \right] \ for\ S(\vec{s}_2, \vec{r}_2) < 0, \Delta_2^+ \geq 0 \tag{40}$$

### 3.4. Time complexity

In the presented algorithm variables are iterating within a rectangle. Each pair of the variables($r_k$,$s_k$), $k = 2,3$ is placed on the area proportional to the upper constraint $N$ of the considered domain (6) of solving equation (3). The remaining variable $s_1$ iterates within the ranges (39) or (40), depending on the sign of the $S(\vec{s}_2, \vec{r}_2)$ (20). So the time complexity of the presented algorithm can be determined as follows by (41):

$$\begin{aligned} T(N) &= O(N^4) \cdot O\left( N + \sqrt{N^2 + \frac{S(\vec{s}_2, \vec{r}_2)}{m_2 m_3}} - 1 - \sqrt{1 + \frac{S(\vec{s}_2, \vec{r}_2)}{m_2 m_3}} \right) = \\ &= O(N^4) \cdot O(N) = O(N^5). \end{aligned} \tag{41}$$

Summarising, the time performance is improved by the $O(N)$ term compared to the brute-force algorithm.

## 4. Improved Brute Force Algorithm

Let us consider the given equation in the form (42):

$$j_1^2 - i_1^2 = \frac{m_1}{m_2}\left(i_2^2 - j_2^2\right) + \frac{m_1}{m_3}\left(i_3^2 - j_3^2\right) \tag{42}$$

**Step 1**: Let us iterate through all the possible 5-ths $(i_1, i_2, i_3, j_2, j_3)$ all the steps 2-3.

**Step 2**: Calculate the right side of the equation (42)

$$x = \frac{m_1}{m_2}\left(i_2^2 - j_2^2\right) + \frac{m_1}{m_3}\left(i_3^2 - j_3^2\right) \tag{43}$$

**Step 3**: From (42) we have:

$$j_1^2 - i_1^2 = x \tag{44}$$

The remaining $j_1$ variable can be calculated from (45):

$$j_1 = \begin{cases} \sqrt{x + i_1^2} \ \ for\, x + i_1^2 > 0 \\ \sqrt{-\left(x + i_1^2\right)} \ \ for\, x + i_1^2 < 0 \end{cases} \tag{45}$$

**Step 4**: If $j_1 \in Z$ add the $\left(i_1, i_2, i_3, j_1, j_2, j_3\right)$ to the set of the solutions.

It is obvious that the time complexity of the improved brute force algorithm is $O\left(N^5\right)$.

## 5. Example

Let us consider the following parameters (46) in the equation (3):

$$m_1 = 1, \quad m_2 = 3, \quad m_3 = 5 \tag{46}$$

The time of the execution of the considered algorithm and the brute-force algorithm is presented in the figure below:

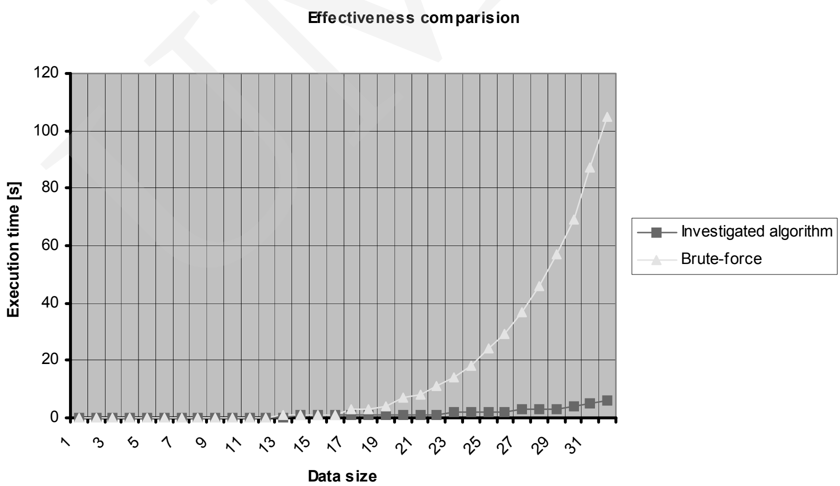


Fig. 1

## 6. Summary

As follows from the example the effectiveness of the improved brute-force algorithm and the algorithm presented in this article is comparable. But the improved brute-force algorithm requires calculating the square roots. The proposed algorithm requires only calculating the four basic operators by the machine and works within the integer arithmetic.

## References

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