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Difference schemes of arbitrary order of accuracy for semilinear parabolic equations

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Abstract – The Cauchy problem for a semilinear parabolic equation is considered. Under the conditions $u(x,t) = X(x)T_1(t) + T_2(t)$, $\frac{\partial u}{\partial x} \neq 0$, it is shown that the problem is equivalent to the system of two ordinary differential equations for which exact difference scheme (EDS) with special Steklov averaging and difference schemes with arbitrary order of accuracy (ADS) are constructed on the moving mesh. The special attention is paid to investigating approximation, stability and convergence of the ADS. The convergence of the iteration method is also considered. The presented numerical examples illustrate theoretical results investigated in the paper.

1 Introduction

In which cases an EDS or an ADS approximating nonlinear parabolic equation can be constructed? The paper deals with this question. The simple technique is presented and the main features of the constructed scheme are considered.

Definition 2. [3] A difference scheme is exact if the truncation error is equal to zero or the exact solution agrees with the numerical solution at the grid nodes.

The problem of constructing a difference scheme of high order of accuracy is topical. In papers [1, 6] the EDSs and truncated difference schemes of an arbitrary rank were constructed for the nonlinear second order differential equation and for the systems of first-order nonlinear

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ordinary differential equations, respectively. It is worth mentioning here the paper by Mickens [9] in which nonstandard finite difference schemes are introduced. In [10] the investigations of the order of approximation, stability, and convergence of the high accuracy difference schemes for the nonlinear transfer equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f(u)$ have been made. The EDS and the difference schemes of an arbitrary order of approximation for the parabolic equations with travelling wave solutions $u(x,t) = U(x - at)$ were constructed in [7, 8].

It was earlier established that for the problems in the parabolic equations with separated variables solutions $u(x,t) = X(x)T_1(t) + T_2(t)$ the EDS and the ADSs may be constructed [3, 7]. The main aim of this paper is to investigate approximation, stability and convergence of the nonlinear scheme of an arbitrary order of accuracy.

Let $C_n^m(\overline{Q}_T)$ be the class of functions with m derivatives in x and n derivatives in t, all derivatives being continuous in the domain $\overline{Q}_T$, where $\overline{Q}_T = \mathbb{R} \times [0,T]$.

Let us consider the Cauchy problem for the semilinear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_1(x)k_2(t)\frac{\partial u}{\partial x}\right) + f(u),\ x \in \mathbb{R},\ 0 < t \le T, \tag{1}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R}. \tag{2}$$

Under the conditions $u(x,t) = X(x)T_1(t) + T_2(t)$, $\frac{\partial u}{\partial x} \neq 0$, we show that problem (1)-(2) is equivalent to the system of two ordinary differential equations [7]

$$\frac{dx}{dt} = c(x)k_2(t),\ x(0) \in \mathbb{R}, \quad c(x) = -\frac{(k_1(x)u_0'(x))'}{u_0'(x)}, \tag{3}$$

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=c(x)k_2(t)} = f(u),\ u(x(0),0) = u_0(x(0)). \tag{4}$$

From (3) we find the curve $x = x(t)$, along which we get from (4) the solution $u(x,t) = u(x(t),t)$ of the problem (1)-(2). Here $x(0) = x^0 \in \mathbb{R}$ is the initial state of the curve $x = x(t)$.

The ADS applying the trapezoidal rule is proposed in the case when the integrals in the EDS cannot be evaluated exactly. We prove that the orders of approximation and accuracy depend only on time step τ divided by the natural constant $m \ge 1$. We also consider the convergence of the iteration method which is used to find the solution of the nonlinear scheme. At the end we present the results of the numerical experiments which illustrate theoretical results stated in the paper. We also construct a difference scheme of an arbitrary order of accuracy applying the Euler-Maclaurin formula instead of the trapezoidal rule [2]. We investigate numerically that the error of this method equals $O\left(\left(\frac{\tau}{m}\right)^{2M+2}\right)$, where $m,\ M$ are positive, natural constants.

2 Statement of the problem

Let us investigate in the domain $\overline{Q}_T$ the Cauchy problem for the semilinear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_1(x)k_2(t)\frac{\partial u}{\partial x}\right) + f(u),\ x \in \mathbb{R},\ 0 < t \le T, \tag{5}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R},\ \sup_{x\in\mathbb{R}} |u_0| < \infty, \tag{6}$$

$$u(x,t) = X(x)T_1(t) + T_2(t),\ \frac{\partial u}{\partial x} \neq 0,\ (x,t) \in \overline{Q}_T, \tag{7}$$

$$0 < m_1 \le k_1(x)k_2(t) \le m_2, (x,t) \in \overline{Q}_T,\ m_1, m_2 = const,\ f(u) \neq 0,\ u \in \mathbb{R}. \tag{8}$$

Let us assume that problem (5) - (8) has a unique solution $u \in C_1^4\left(\overline{Q}_T\right)$ with all derivatives bounded and $k_1(x) \in C^3(\mathbb{R}),\ k_2(t) \in C^2([0,T]),\ f(u) \in C^2(\mathbb{R})$. Assumption (7) is the key relation. Taking into account these assumptions, we rewrite problem (5)-(6) in the following form

$$\frac{dx}{dt} = -\frac{\frac{\partial}{\partial x}\left(k_1(x)k_2(t)\frac{\partial u}{\partial x}\right)}{\frac{\partial u}{\partial x}} = -\frac{(k(x)u_0'(x))'}{u_0'(x)}k_2(t) = c(x)k_2(t),\ x(0) \in \mathbb{R}, \tag{9}$$

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=c(x)k_2(t)} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial t} = f(u),\ u(x(0),0) = u_0(x(0)). \tag{10}$$

We assume that $c(x) \neq 0,\ x \in \mathbb{R},\ c(x) \in C^2(\mathbb{R})$.

3 The difference scheme of an arbitrary order of accuracy

In this Section, the EDS approximating problem (5) - (8) is considered. The ADS is constructed in the case when the integrals in the EDS cannot be evaluated exactly. The trapezoidal rule is applied to approximate the integrals.

Let us introduce space and time grids

$$\overline{\omega}_h^0 = \left\{x_i^0 = ih_i^0,\ i \in \mathbb{Z}\right\},\ \overline{\omega}_\tau = \left\{t_n = n\tau,\ n = \overline{0, N_0},\ \tau = \frac{T}{N_0}\right\},$$

here $h_i^n = x_{i+1}^n - x_i^n$ is the space step at time $t = t_n$.

In [3, 7] the EDS approximating problem (5) - (8) was constructed

$$\frac{x_i^{n+1} - x_i^n}{\tau} = \left(\frac{1}{x_i^{n+1} - x_i^n}\int_{x_i^n}^{x_i^{n+1}} \frac{dx}{c(x)}\right)^{-1} \frac{1}{\tau}\int_{t_n}^{t_{n+1}} k_2(t)dt, \qquad x_i^0 \in \overline{\omega}_h^0,\ i \in \mathbb{Z},\ n = \overline{0, N_0 - 1}, \tag{11}$$

$$\frac{y_i^{n+1} - y_i^n}{\tau} = \left(\frac{1}{y_i^{n+1} - y_i^n}\int_{y_i^n}^{y_i^{n+1}} \frac{du}{f(u)}\right)^{-1},\ y_i^0 = u_0(x_i^0),\ i \in \mathbb{Z},\ n = \overline{0, N_0 - 1}. \tag{12}$$

Only in some cases the integrals in the EDS can be evaluated exactly. In other cases, applying the trapezoidal rule, we approximate problem (9) - (10) by the difference scheme

$$\frac{x_{hi}^{n+1}-x_{hi}^{n}}{\tau}=\varphi_1\left(x_{hi}^{n},x_{hi}^{n+1}\right)\varphi_2\left(t_n,t_{n+1}\right),\ x_{hi}^{0}=x_i^0\in\overline{\omega}_h^0,\ i\in\mathbb{Z},\ n=\overline{0,N_0-1}, \quad (13)$$

$$\frac{y_i^{n+1}-y_i^{n}}{\tau}=\varphi_3\left(y_i^{n},y_i^{n+1}\right),\ y_i^0=u_0(x_{hi}^0),\ i\in\mathbb{Z},\ n=\overline{0,N_0-1}, \quad (14)$$

where $m\geq 1$ is a natural number and $\varphi_1,\varphi_2,\varphi_3:\mathbb{R}^2\to\mathbb{R}$ are introduced as follows

$$\varphi_1(s_1,s_2)=\left[\frac{1}{2m}\left(\frac{1}{c(s_1)}+\frac{1}{c\left(s_2\right)}\right)+\frac{1}{m}\sum_{j=1}^{m-1}\frac{1}{c\left(s_1+j\frac{s_2-s_1}{m}\right)}\right]^{-1},$$

$$\varphi_2(s_1,s_2)=\left(k_2(s_1)+k_2\left(s_2\right)\right)/(2m)+\sum_{j=1}^{m-1}k_2\left(s_1+j\frac{s_2-s_1}{m}\right)/m,$$

$$\varphi_3(s_1,s_2)=\left[\frac{1}{2m}\left(\frac{1}{f(s_1)}+\frac{1}{f\left(s_2\right)}\right)+\frac{1}{m}\sum_{j=1}^{m-1}\frac{1}{f\left(s_1+j\frac{s_2-s_1}{m}\right)}\right]^{-1}.$$

Let us denote the approximate value of the curve $x_i(t)$ on $n-th$ level by x_{hi}^n, and the exact value of this curve on $n-th$ level by x_i^n. The curve $x_i(t)$ is the solution of problem (9) with the initial value x_i^0. Here y_i^n is the approximate solution of problem (10) in the node (x_{hi}^n,t_n) and $u_i^n,\ u_{hi}^n$ are the values of exact solution of this problem in the nodes (x_i^n,t_n) and (x_{hi}^n,t_n), respectively.

Equation (13) introduces the moving grid in the domain $\overline{Q}_T$

$$\overline{\omega}=\left\{(x_{hi}^n,t_n)\in\overline{Q}_T:\ x_{hi}^0=ih_i^0,\ i\in\mathbb{Z},\ t_n=n\tau,\ n=\overline{0,N_0}\right\}.$$

Let us denote the error of the method by

$$\delta x_{hi}^n=x_{hi}^n-x_i^n,\ \delta u_{hi}^n=y_i^n-u_{hi}^n.$$

Then the difference problem for the error of the method is written in the following form

$$\begin{gathered}
\frac{\delta x_{hi}^{n+1}-\delta x_{hi}^{n}}{\tau}-\delta x_{hi}^{n}\varphi_2(t_n,t_{n+1})\frac{\partial\varphi_1}{\partial s_1}\left(x_{\theta 1i}^{n},x_{\theta 2i}^{n+1}\right)\\
-\delta x_{hi}^{n+1}\varphi_2(t_n,t_{n+1})\frac{\partial\varphi_1}{\partial s_2}\left(x_{\theta 1i}^{n},x_{\theta 2i}^{n+1}\right)=\psi_{1i}^{n},\\
\delta x_{hi}^{0}=0,\ i\in\mathbb{Z},\ n=\overline{0,N_0-1},\\
\frac{\delta u_{hi}^{n+1}-\delta u_{hi}^{n}}{\tau}-\delta u_{hi}^{n}\frac{\partial\varphi_3}{\partial s_1}(u_{\theta 3i}^{n},u_{\theta 4i}^{n+1})-\delta u_{hi}^{n+1}\frac{\partial\varphi_3}{\partial s_2}(u_{\theta 3i}^{n},u_{\theta 4i}^{n+1})\\
+\delta x_{hi}^{n+1}\frac{\partial u}{\partial x}(x_{\theta 6i}^{n+1},t_{n+1})\left(\frac{1}{\tau}-\frac{\partial\varphi_3}{\partial s_2}(u_{\theta 3i}^{n},u_{\theta 4i}^{n+1})\right)\\
-\delta x_{hi}^{n}\frac{\partial u}{\partial x}(x_{\theta 5i}^{n},t_n)\left(\frac{1}{\tau}+\frac{\partial\varphi_3}{\partial s_1}(u_{\theta 3i}^{n},u_{\theta 4i}^{n+1})\right)=\psi_{2i}^{n},\\
\delta u_{hi}^{0}=0,\ i\in\mathbb{Z},\ n=\overline{0,N_0-1},
\end{gathered}\tag{15}$$

where $x_{\theta pi}^{n}=x_i^n+\theta_{pi}^{n}\delta x_{hi}^{n},\ u_{\theta pi}^{n}=u_i^n+\theta_{pi}^{n}\left(\delta u_{hi}^{n}+u_{hi}^{n}-u_i^n\right)$. Here and after $\theta_{pi}^{n}=const,\ 0<\theta_{pi}^{n}<1$. The approximation error of the difference scheme (13) - (14) equals

$$\psi_i^n=\psi_{1i}^n+\psi_{2i}^n,$$

where $\psi_{1i}^{n}=-\frac{x_{hi}^{n+1}-x_{hi}^{n}}{\tau}+\varphi_1\left(x_i^n,x_i^{n+1}\right)\varphi_2(t_n,t_{n+1}),\quad \psi_{2i}^{n}=-\frac{u_i^{n+1}-u_i^n}{\tau}+\varphi_3(u_i^n,u_i^{n+1})$.

4 Approximation

In this Section, we prove the theorem on an order of approximation of difference scheme (13) - (14).

Theorem 2. Suppose that

$$c(x)\in C^2(\mathbb{R}),\ 0<c_1\le\left|\frac{1}{c(x)}\right|\le c_2,\ \left|\left(\frac{1}{c(x)}\right)^{(2)}\right|\le c_3,\ x\in\mathbb{R},\tag{16}$$

$$k_2(t)\in C^2([0,T]),\ |k_2(t)|\le c_4,\ |k_2''(t)|\le c_5,\ t\in[0,T],\tag{17}$$

$$f(u)\in C^2(\mathbb{R}),\ 0<c_6\le\left|\frac{1}{f(u)}\right|\le c_7,\ \left|\left(\frac{1}{f(u)}\right)^{(2)}\right|\le c_8,\ u\in\mathbb{R},\tag{18}$$

$$\left\|\frac{\partial u}{\partial t}\right\|_{C(\overline{Q}_T)}\le c_9,\tag{19}$$

where constants $c_p>0$. Then difference scheme (13)-(14) approximates problem (9)-(10) with the truncation error equal to $O\left(\left(\frac{\tau}{m}\right)^2\right)$.

PROOF. Our first goal is to show that $\psi_{1i}^n = R_i^n\left(\frac{\tau}{m}\right)^2$, where the expression $|R_i^n|$ is bounded. It is straightforward to show that the approximation error ψ_{1i}^n satisfies the chain of relations

$$\psi_{1i}^n = \left[\frac{-1}{x_i^{n+1}-x_i^n}\int\limits_{x_i^n}^{x_i^{n+1}}\frac{dx}{c(x)}\right]^{-1}\frac{1}{\tau}\int\limits_{t_n}^{t_{n+1}}k_2(t)dt + \varphi_1\left(x_i^n, x_i^{n+1}\right)\varphi_2(t_n, t_{n+1})$$

$$= -\frac{\frac{x_i^{n+1}-x_i^n}{m}\left[\frac{1}{2c(x_i^n)}+\frac{1}{2c(x_i^{n+1})}+\sum\limits_{j=1}^{m-1}\frac{1}{c\left(x_i^n+j\frac{x_i^{n+1}-x_i^n}{m}\right)}\right]-\int\limits_{x_i^n}^{x_i^{n+1}}\frac{dx}{c(x)}}{\left[\int\limits_{x_i^n}^{x_i^{n+1}}\frac{dx}{c(x)}\right]\left[\varphi_1\left(x_i^n, x_i^{n+1}\right)\right]^{-1}}\frac{1}{\tau}\int\limits_{t_n}^{t_{n+1}}k_2(t)dt$$

$$+\frac{1}{12}\left(\frac{\tau}{m}\right)^2 k_2''(t_{\theta 7n})\varphi_1(x_i^n, x_i^{n+1}) = \left(\frac{\tau}{m}\right)^2 R_i^n,$$

where $i\in\mathbb{Z},\ n=\overline{0, N_0-1}$ and

$$R_i^n = -\frac{\frac{x_i^{n+1}-x_i^n}{12}\left(\frac{dx(t_{\theta 8n})}{dt}\right)^2\left(\frac{1}{c(x_{\theta 9i})}\right)^{(2)}}{\left[\int\limits_{x_i^n}^{x_i^{n+1}}\frac{dx}{c(x)}\right]\left[\varphi_1\left(x_i^n, x_i^{n+1}\right)\right]^{-1}}\frac{1}{\tau}\int\limits_{t_n}^{t_{n+1}}k_2(t)dt + \frac{1}{12}k_2''(t_{\theta 7n})\varphi_1(x_i^n, x_i^{n+1}),$$

$$x_{\theta 9i} = x_i^n + \theta_{9i}^n\left(x_i^{n+1}-x_i^n\right),\ t_{\theta 7n}, t_{\theta 8n}\in(t_n, t_{n+1}).$$

Since (16), the function $\frac{1}{c(x)}$ has a constant sign for all $x\in\mathbb{R}$. Thus, the following estimate is valid

$$\left|\left[\varphi_1\left(x_i^n, x_i^{n+1}\right)\right]^{-1}\right| = \frac{1}{m}\left\{\left|\frac{1}{2c(x_i^n)}\right| + \left|\frac{1}{2c\left(x_i^{n+1}\right)}\right| + \sum_{j=1}^{m-1}\left|\frac{1}{c\left(x_i^n + j\frac{x_i^{n+1}-x_i^n}{m}\right)}\right|\right\} \geq c_1.$$

Some tedious manipulations yields

$$|R_i^n| \leq M_A,\ M_A = \left(c_1^{-4}c_3c_4^3 + c_1^{-1}c_5\right)/12.$$

Thus, difference equation (13) approximates differential problem (9) with the second order with respect to $\frac{\tau}{m}$

$$\max_{0\leq n<N_0}\|\psi_1^n\|_C \leq M_A\left(\frac{\tau}{m}\right)^2,\ \|\psi_1^n\|_C = \sup_{i\in\mathbb{Z}}|\psi_{1i}^n|.$$

Analogously, we can show that difference equation (14) approximates differential problem (10) with the second order with respect to $\frac{\tau}{m}$

$$\max_{0\leq n<N_0}\|\psi_2^n\|_C \leq M_A'\left(\frac{\tau}{m}\right)^2,\ M_A' = c_6^{-2}c_8c_9^2/12.$$

Finally, we find that difference scheme (13) - (14) approximates differential problem (9) - (10) with the second order with respect to $\frac{\tau}{m}$

$$\max_{0\le n<N_0} \|\psi^n\|_C \le (M_A + M_A') \left(\frac{\tau}{m}\right)^2 .$$

□

5 Stability

In this Section, stability of the ADS is investigated. Let us perturb the initial data of problem (13) - (14)

$$\frac{\tilde{x}_{hi}^{n+1} - \tilde{x}_{hi}^{n}}{\tau} = \varphi_1\left(\tilde{x}_{hi}^{n}, \tilde{x}_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1}),\ \tilde{x}_{hi}^{0} = \tilde{x}_i^0,\ i\in\mathbb{Z},\ n = \overline{0, N_0-1}, \tag{20}$$

$$\frac{\tilde{y}_i^{n+1} - \tilde{y}_i^{n}}{\tau} = \varphi_3\left(\tilde{y}_i^{n}, \tilde{y}_i^{n+1}\right),\ \tilde{y}_i^0 = \tilde{u}_0(\tilde{x}_{hi}^0),\ i\in\mathbb{Z},\ n = \overline{0, N_0-1}. \tag{21}$$

We give a theorem on stability of difference scheme (13)-(14).

Theorem 3. Suppose that

$$c(x)\in C^1(\mathbb{R}),\ 0 < c_1 \le \left|\frac{1}{c(x)}\right| \le c_2,\ \left|\left(\frac{1}{c(x)}\right)^{(1)}\right| \le c_{10},\ x\in\mathbb{R}, \tag{22}$$

$$k_2(t)\in C^1([0,T]),\ |k_2(t)|\le c_4,\ t\in[0,T], \tag{23}$$

$$f(u)\in C^1(\mathbb{R}),\ 0 < c_6 \le \left|\frac{1}{f(u)}\right| \le c_7,\ \left|\left(\frac{1}{f(u)}\right)^{(1)}\right| \le c_{11},\ u\in\mathbb{R}, \tag{24}$$

where the constants $c_p > 0$. Then for sufficiently small $\tau \le \min\{\tau_0^*, \tau_0^{**}\}$ difference scheme (13)-(14) is stable with respect to small perturbation of the initial values and the following estimates hold

$$\max_{0\le n\le N_0} \|\tilde{x}_h^n - x_h^n\|_C \le e^{T(M_2+2M_1)} \left\|\tilde{x}_h^0 - x_h^0\right\|_C,\ M_1, M_2 = const > 0,$$
$$\max_{0\le n\le N_0} \|\tilde{y}^n - y^n\|_C \le e^{T(M_4+2M_3)} \|\tilde{u}_0 - u_0\|_C,\ M_3, M_4 = const > 0.$$

PROOF. Under assumptions (22), (24), the functions $\varphi_1, \varphi_3 \in C^1(\mathbb{R}^2)$ have bounded derivatives

$$\left|\frac{\partial\varphi_1}{\partial s_2}(s, s_2)\right| \le M_1,\ \left|\frac{\partial\varphi_1}{\partial s_1}(s_1, s_2)\right| \le M_2,\ (s_1, s_2)\in\mathbb{R}^2,$$
$$\left|\frac{\partial\varphi_3}{\partial s_2}(s, s_2)\right| \le M_3,\ \left|\frac{\partial\varphi_3}{\partial s_1}(s_1, s_2)\right| \le M_4,\ (s_1, s_2)\in\mathbb{R}^2,$$

where M_p are positive constants.

Subtracting equations (13) - (14) from (20) - (21) and introducing the notation $\Delta x_{hi}^n = \tilde{x}_{hi}^n - x_{hi}^n$, $\Delta y_i^n = \tilde{y}_i^n - y_i^n$, we get

$$\frac{\Delta x_{hi}^{n+1} - \Delta x_{hi}^n}{\tau} = \left(\Delta x_{hi}^n \frac{\partial \varphi_1}{\partial s_1}\left(x_{\theta 10i}^n, x_{\theta 11i}^{n+1}\right) + \Delta x_{hi}^{n+1} \frac{\partial \varphi_1}{\partial s_2}\left(x_{\theta 10i}^n, x_{\theta 11i}^{n+1}\right)\right) \varphi_2(t_n, t_{n+1}),$$
$$\Delta x_{hi}^0 = \tilde{x}_i^0 - x_i^0,\ i \in \mathbb{Z},\ n = \overline{0, N_0 - 1},$$
$$\frac{\Delta y_i^{n+1} - \Delta y_i^n}{\tau} = \Delta y_i^n \frac{\partial \varphi_3}{\partial s_1}\left(y_{\theta 12i}^n, y_{\theta 13i}^{n+1}\right) + \Delta y_i^{n+1} \frac{\partial \varphi_3}{\partial s_2}\left(y_{\theta 12i}^n, y_{\theta 13i}^{n+1}\right),$$
$$\Delta y_i^0 = \tilde{u}_0(\tilde{x}_{hi}^0) - u_0(x_{hi}^0),\ i \in \mathbb{Z},\ n = \overline{0, N_0 - 1},$$

where $x_{\theta pi}^n = x_{hi}^n + \theta_{pi}^n\left(\tilde{x}_{hi}^n - x_{hi}^n\right),\ y_{\theta pi}^n = y_i^n + \theta_{pi}^n\left(\tilde{y}_i^n - y_i^n\right)$. The following grid node's estimate is valid

$$\left|\Delta x_{hi}^{n+1}\right| \le \tau \left|\frac{\partial \varphi_1}{\partial s_2}\left(x_{\theta 10i}^n, x_{\theta 11i}^{n+1}\right)\right| \left|\varphi_2(t_n, t_{n+1})\right| \left|\Delta x_{hi}^{n+1}\right| +$$
$$+ \left(1 + \tau \left|\frac{\partial \varphi_1}{\partial s_1}\left(x_{\theta 10i}^n, x_{\theta 11i}^{n+1}\right)\right| \left|\varphi_2(t_n, t_{n+1})\right|\right) \left|\Delta x_{hi}^n\right|$$
$$(1 - \tau M_1 c_4)\left|\Delta x_{hi}^{n+1}\right| \le (1 + \tau M_2 c_4)\left|\Delta x_{hi}^n\right| \le (1 + \tau M_2 c_4)\left\|\Delta x_h^n\right\|_C \le e^{\tau M_2 c_4}\left\|\Delta x_h^n\right\|_C.$$

For sufficiently small $\tau \le \tau_0^*$, where $\tau_0^* = \frac{1}{2M_1 c_4}$, the following inequalities hold

$$\left|\Delta x_{hi}^{n+1}\right| \le \frac{e^{\tau M_2 c_4}}{1 - \tau M_1 c_4}\left\|\Delta x_h^n\right\|_C \le e^{\tau(M_2 + 2M_1)c_4}\left\|\Delta x_h^n\right\|_C.$$

Thus

$$\left\|\Delta x_h^{n+1}\right\|_C \le e^{\tau(M_2 + 2M_1)c_4}\left\|\Delta x_h^n\right\|_C \le \dots \le e^{(n+1)\tau(M_2 + 2M_1)c_4}\left\|\Delta x_h^0\right\|_C.$$

Because it is satisfied for any $n = \overline{0, N_0 - 1}$, then

$$\max_{0 \le n \le N_0}\left\|\Delta x_h^n\right\|_C \le e^{T(M_2 + 2M_1)c_4}\left\|\Delta x_h^0\right\|_C. \tag{25}$$

Analogously, we can prove that for sufficiently small $\tau \le \tau_0^{**}$, where $\tau_0^{**} = \frac{1}{2M_3}$

$$\max_{0 \le n \le N_0}\left\|\Delta y^n\right\|_C \le e^{T(M_4 + 2M_3)}\left\|\Delta u_0\right\|_C. \tag{26}$$

Estimates (25), (26) express the stability of difference scheme (13) - (14) with respect to the small perturbation of the initial values. □

6 Convergence

In this Section, we prove the theorem on an order of accuracy of difference scheme (13) - (14).

Theorem 4. Let the assumptions of Theorems 2 and 3 holds. Then for sufficiently small $\tau \leq \min\{\tau_0^*, \tau_0^{**}\}$ the solution of the difference scheme (13) - (14) converges to the solution of the differential problem (9) - (10) and the estimates hold

$$\max_{0\leq n\leq N_0} \|x_h^n - x^n\|_C \leq M_Z \left(\frac{\tau}{m}\right)^2, \; M_Z = const > 0,$$
$$\max_{0\leq n\leq N_0} \|y^n - u_h^n\|_C \leq M_Z' \left(\frac{\tau}{m}\right)^2, \; M_Z' = const > 0.$$

PROOF. Similarly to that before, the functions $\varphi_1, \varphi_3 \in C^1(\mathbb{R}^2)$ have bounded derivatives

$$\left|\frac{\partial \varphi_1}{\partial s_2}(s, s_2)\right| \leq M_1, \; \left|\frac{\partial \varphi_1}{\partial s_1}(s_1, s_2)\right| \leq M_2, \; (s_1, s_2) \in \mathbb{R}^2,$$
$$\left|\frac{\partial \varphi_3}{\partial s_2}(s, s_2)\right| \leq M_3, \; \left|\frac{\partial \varphi_3}{\partial s_1}(s_1, s_2)\right| \leq M_4, \; (s_1, s_2) \in \mathbb{R}^2,$$

where M_p are positive constants. From the equation for the error of the method the following estimates hold

$$\left|\delta x_{hi}^{n+1}\right| \leq \tau \left|\frac{\partial \varphi_1}{\partial s_2}\left(x_{\theta 1i}^n, x_{\theta 2i}^{n+1}\right)\right| \left|\varphi_2(t_n, t_{n+1})\right| \left|\delta x_{hi}^{n+1}\right| +$$
$$+ \left|1 + \tau \frac{\partial \varphi_1}{\partial s_1}\left(x_{\theta 1i}^n, x_{\theta_2 i}^{n+1}\right)\right| \left|\varphi_3(t_n, t_{n+1})\right| \left|\delta x_{hi}^n\right| + \tau \left|\psi_{1i}^n\right|$$
$$(1 - \tau M_1 c_4)\left|\delta x_{hi}^{n+1}\right| \leq (1 + \tau M_2 c_4)\left\|\delta x_h^n\right\|_C + \tau \left\|\psi_1^n\right\|_C.$$

For sufficiently small $\tau \leq \tau_0^*$, where $\tau_0^* = \frac{1}{2M_1 c_4}$, we have

$$\left|\delta x_{hi}^{n+1}\right| \leq \frac{e^{\tau M_2 c_4}}{1 - \tau M_1 c_4}\left\|\delta x_h^n\right\|_C + \frac{\tau}{1 - \tau M_1 c_4}\left\|\psi_1^n\right\|_C \leq$$
$$\leq e^{\tau(M_2 + 2M_1)c_4}\left\|\delta x_h^n\right\|_C + \tau e^{2\tau M_1 c_4}\left\|\psi_1^n\right\|_C.$$

The above inequalities are valid for any $i \in \mathbb{Z}$, thus

$$\left\|\delta x_h^{n+1}\right\|_C \leq e^{\tau(M_2 + 2M_1)c_4}\left\|\delta x_h^n\right\|_C + \tau e^{2\tau M_1 c_4}\left\|\psi_1^n\right\|_C \leq$$
$$\leq \cdots \leq e^{(n+1)\tau(M_2 + 2M_1)c_4}\left\|\delta x_h^0\right\|_C + \tau \sum_{k=0}^{n} e^{(n-k)\tau(M_2 + 2M_1)c_4 + 2\tau M_1 c_4}\left\|\psi_1^k\right\|_C \leq$$
$$\leq \max_{0\leq k\leq n}\left\|\psi_1^k\right\|_C t_{n+1} e^{T(M_2 + 4M_1)c_4}.$$

Since $\psi_{1i}^n = O\left(\left(\frac{\tau}{m}\right)^2\right)$, then

$$\max_{0\leq n\leq N_0}\left\|\delta x_h^n\right\|_C \leq M_Z \left(\frac{\tau}{m}\right)^2, \; M_Z = e^{T(M_2 + 4M_1)c_4} T M_A. \tag{27}$$

To obtain the estimation of the error δu_{hi}^n a little manipulation is needed

$$\left\|\delta u_h^{n+1}\right\|_C \leq \left\|\delta y^{n+1}\right\|_C + \left\|u^{n+1} - u_h^{n+1}\right\|_C \leq \left\|\delta y^{n+1}\right\|_C + \left\|\delta x_h^{n+1}\right\|_C \left\|\frac{\partial u}{\partial x}\right\|_{C(\overline{Q}_T)},$$

where $\delta y_i^n = y_i^n - u_i^n$ satisfies the equation

$$\frac{\delta y_i^{n+1} - \delta y_i^n}{\tau} - \delta y_i^n \frac{\partial \varphi_3}{\partial s_1}\left(y_{\theta 14i}^n, y_{\theta 15i}^{n+1}\right) - \delta y_i^{n+1} \frac{\partial \varphi_3}{\partial s_2}\left(y_{\theta 14i}^n, y_{\theta 15i}^{n+1}\right) = \psi_{2i}^n,$$
$$y_{\theta 14i}^n = u_i^n + \theta_{14i}^n \delta y_i^n,\ y_{\theta 15i}^{n+1} = u_i^{n+1} + \theta_{14i}^{n+1} \delta y_i^{n+1}.$$

Our problem reduces to estimating the error δy_i^n. A reasoning similar to that used in the first part of the proof, shows that

$$\left\|\delta y^{n+1}\right\|_C \leq \max_{0\leq k\leq n} \left\|\psi_2^k\right\|_C t_{n+1} e^{T(M_4+4M_3)}.$$

We get

$$\max_{0\leq n\leq N_0} \|\delta u_h^n\|_C \leq T e^{T(M_4+4M_3)} \max_{0\leq k\leq n} \left\|\psi_2^k\right\|_C$$
$$+ T e^{T(M_2+4M_1)c_4} \left\|\frac{\partial u}{\partial x}\right\|_{C(\overline{Q}_T)} \max_{0\leq k\leq n} \left\|\psi_1^k\right\|_C \leq M_Z' \left(\frac{\tau}{m}\right)^2,\ M_Z'$$
$$= T e^{T(M_4+4M_3)} M_A' + T e^{T(M_2+4M_1)c_4} M_A \left\|\frac{\partial u}{\partial x}\right\|_{C(\overline{Q}_T)}. \tag{28}$$

Finally, estimates (27) and (28) express the convergence of the solution of difference scheme (13) - (14) to the solution of differential problem (9) - (10) with the second order with respect to $\frac{\tau}{m}$. □

7 Iteration method

In this Section, convergence of the iteration method for difference scheme (13) - (14) is discussed.

The following iteration method is used in connection with solving nonlinear difference scheme (13) - (14)

$$\frac{\overset{s+1}{x}{}_{hi}^{n+1} - x_{hi}^n}{\tau} = \varphi_1\left(x_{hi}^n, \overset{s}{x}{}_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1}),$$
$$\overset{0}{x}{}_{hi}^{n+1} = x_{hi}^n + \tau c(x_{hi}^n) k_2(t_n),\ x_{hi}^0 = x_i^0 \in \omega_h^0,\ n = \overline{0, N_0 - 1}, \tag{29}$$

$$\frac{\overset{s+1}{y}{}_i^{n+1} - y_i^n}{\tau} = \varphi_3\left(y_i^n, \overset{s}{y}{}_i^{n+1}\right),$$
$$\overset{0}{y}{}_i^{n+1} = y_i^n + \tau f(y_i^n),\ y_i^0 = u_0(x_{hi}^0),\ i \in \mathbb{Z},\ n = \overline{0, N_0 - 1}. \tag{30}$$

We give a theorem on convergence of iteration method (29) - (30).

Theorem 5. Let the assumptions of Theorem 4 holds. Then for sufficiently small $\tau \leq \tau_0,\ \tau_0 \leq \min\left\{\frac{q}{M_1 c_4}, \frac{q}{M_3}\right\},\ 0<q<1$ the solution of iteration method (29) - (30) converges to the solution of difference equation (13) - (14).

PROOF. Let τ be chosen sufficiently small

$$\tau \leq \tau_0,\ \tau_0 \leq \min\left\{\frac{q}{M_1 c_4}, \frac{q}{M_3}\right\},\ 0<q<1.$$

First, we estimate the difference between $\overset{1}{x}{}_{hi}^{n+1}$ and the initial approximation

$$\begin{aligned}\left|\overset{1}{x}{}_{hi}^{n+1} - \overset{0}{x}{}_{hi}^{n+1}\right| &= \left|x_{hi}^{n} + \tau\varphi_1\left(x_{hi}^{n}, \overset{0}{x}{}_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1}) - x_{hi}^{n} - \tau c(x_{hi}^{n})k_2(t_n)\right| \\ &= \tau\left|\varphi_1\left(x_{hi}^{n}, \overset{0}{x}{}_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1}) - c(x_{hi}^{n})k_2(t_n)\right| \leq 2\tau_0 c_1^{-1} c_4.\end{aligned}$$

Now, let us estimate the difference between two successive iterations

$$\begin{aligned}&\left|\overset{s+1}{x}{}_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right| \qquad (31)\\ &= \left|x_{hi}^{n} + \tau\varphi_1\left(x_{hi}^{n}, \overset{s}{x}{}_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1}) - x_{hi}^{n} - \tau\varphi_1\left(x_{hi}^{n}, \overset{s-1}{x}{}_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1})\right| \\ &= \tau\left|\overset{s}{x}{}_{hi}^{n+1} - \overset{s-1}{x}{}_{hi}^{n+1}\right|\left|\frac{\partial\varphi_1}{\partial s_2}\left(x_{hi}^{n}, x_{\theta_{16i}}^{n+1}\right)\right|\left|\varphi_2(t_n, t_{n+1})\right| \leq \tau M_1 c_4\left|\overset{s}{x}{}_{hi}^{n+1} - \overset{s-1}{x}{}_{hi}^{n+1}\right| \\ &\leq q\left|\overset{s}{x}{}_{hi}^{n+1} - \overset{s-1}{x}{}_{hi}^{n+1}\right| \leq \dots \leq q^s\left|\overset{1}{x}{}_{hi}^{n+1} - \overset{0}{x}{}_{hi}^{n+1}\right| \leq 2\tau_0 c_1^{-1} c_4 q^s.\end{aligned}$$

where $x_{\theta_{16i}}^{n+1} = \overset{s-1}{x}{}_{hi}^{n+1} + \theta_{16i}^{n+1}\left(\overset{s}{x}{}_{hi}^{n+1} - \overset{s-1}{x}{}_{hi}^{n+1}\right)$. We are now in a position to estimate the error $x_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}$

$$\begin{aligned}\left|x_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right| &\leq \left|x_{hi}^{n+1} - x_{hi}^{n} - \tau\varphi_1\left(x_{hi}^{n}, x_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1})\right| \\ &+ \left|x_{hi}^{n} + \tau\varphi_1\left(x_{hi}^{n}, x_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1}) - x_{hi}^{n} - \tau\varphi_1\left(x_{hi}^{n}, \overset{s}{x}{}_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1})\right| \\ &+ \left|x_{hi}^{n} + \tau\varphi_1\left(x_{hi}^{n}, \overset{s}{x}{}_{hi}^{n+1}\right)\varphi_2(t_n, t_{n+1}) - \overset{s}{x}{}_{hi}^{n+1}\right| \\ &\leq \tau\left|x_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right|\left|\frac{\partial\varphi_1}{\partial s_2}\left(x_{hi}^{n}, x_{\theta_{17i}^{n+1}}\right)\right|\left|\varphi_2(t_n, t_{n+1})\right| + \left|\overset{s+1}{x}{}_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right| \\ &\leq q\left|x_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right| + \left|\overset{s+1}{x}{}_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right|.\end{aligned}$$

where $x_{\theta_{17i}}^{n+1} = x_{hi}^{n+1} + \theta_{17i}^{n+1}\left(\overset{s}{x}{}_{hi}^{n+1} - x_{hi}^{n+1}\right)$. Thus we have

$$\left|x_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right| \le \frac{1}{1-q}\left|\overset{s+1}{x}{}_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right| \le M_I q^s,\ M_I = 2\tau_0 c_1^{-1} c_4/(1-q).$$

The above inequality is valid for any $i \in \mathbb{Z}$, so we get

$$\left\|x_h^{n+1} - \overset{s}{x}{}_h^{n+1}\right\|_C \le M_I q^s.$$

It remains to show that

$$\left\|y^{n+1} - \overset{s}{y}{}^{n+1}\right\|_C \le M_I' q^s,\ M_I' = 2\tau_0 c_6^{-1}/(1-q).$$

The proof is similar to that used above.

Thereby the iterations converge with the rate of geometric progression and the limits exist $\lim\limits_{s\to\infty}\left\|x_h^{n+1} - \overset{s}{x}{}_h^{n+1}\right\|_C = 0$, $\lim\limits_{s\to\infty}\left\|y^{n+1} - \overset{s}{y}{}^{n+1}\right\|_C = 0$. □

8 Numerical examples

In this Section, the example of the previously considered ADS is investigated. We also construct the difference scheme with the Euler-Maclaurin formula instead of the trapezoidal rule and investigate numerically that the error of the method equals $O\left(\left(\frac{\tau}{m}\right)^{2M+2}\right)$, where $m,\ M$ are the positive, natural numbers.

Let us consider the boundary-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\ln u,\ 0 < x < l,\ 0 < t \le T,$$

where the boundary and initial conditions are coincidental with the exact solution $u(x,t) = \exp\left\{B_0 e^t + \frac{1}{2} - \frac{x^2}{4}\right\}$, and the function $Q(u) = u\ln u$ is a source for $u > 1$ and a sink for $0 < u < 1$ [12]. We approximate it by the difference scheme

$$\frac{x_i^{n+1} - x_i^n}{\tau} = \left(\frac{1}{x_i^{n+1} - x_i^n}\int\limits_{x_i^n}^{x_i^{n+1}} \frac{dx}{x^2-2}\right)^{-1},\ x_i^0 \in \overline{\omega}_h^0,\ i \in \mathbb{Z},\ n = \overline{0, N_0 - 1}, \tag{32}$$

$$\frac{y_i^{n+1} - y_i^n}{\tau} = \varphi_3\left(y_i^n, y_i^{n+1}\right),\ y_i^0 = u_0(x_{hi}^0),\ i \in \mathbb{Z},\ n = \overline{0, N_0 - 1}. \tag{33}$$

From (32) we find that $x_i^{n+1} = \sqrt{2 + e^\tau\left((x_i^n)^2 - 2\right)}$, where $0 \le t_n \le \ln\frac{2}{2-(x_i^0)^2}$ for $\left|x_i^0\right| < \sqrt{2}$. To solve equation (33) we use iteration method (30). The stopping criterion in the iteration method is

$$\left\|\overset{s+1}{y}{}^{n+1} - \overset{s}{y}{}^{n+1}\right\|_{\overline{C}_{n+1}} \le \epsilon,$$

where ϵ is previously given tolerance. When the above condition is satisfied, we advance to the next level with $y_i^{n+1} = \overset{s+1}{y}{}_i^{n+1}$. Here $\|v^n\|_{\overline{C}_n} = \sup_{i\in P_n} |v_i^n|$, where P^n determines the range of possible values for every index i of the grid nodes on the $n-th$ level.

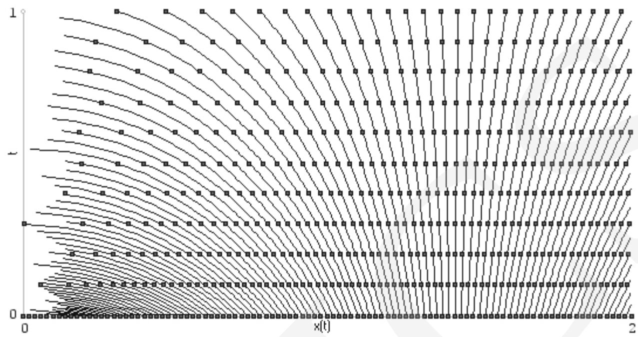

Fig. 1. The moving mesh of scheme (13)-(14).

Table 1 presents the results of the numerical experiments, where S is the number of iterations.

Table 1. $l = 2,\ T = 1,\ \beta_0 = 0.5,\ \epsilon = 1.0 \cdot 10^{-15},\ h_i^0 = 0.02$.

τ	m	$\max_{0\le n\le N_0} \|x_h^n - x(t_n)\|_{\overline{C}_n}$	$\max_{0\le n\le N_0} \|y^n - u_h(t_n)\|_{\overline{C}_n}$	S
0.1	2	$7.73 \cdot 10^{-18}$	$2.01 \cdot 10^{-02}$	17
	10		$8.11 \cdot 10^{-04}$	17
	100		$8.11 \cdot 10^{-06}$	17
	1000		$8.11 \cdot 10^{-08}$	17
0.01	2	$1.60 \cdot 10^{-17}$	$2.01 \cdot 10^{-04}$	8
	10		$8.04 \cdot 10^{-06}$	8
	100		$8.04 \cdot 10^{-08}$	8
	1000		$8.04 \cdot 10^{-10}$	8

Now we investigate numerically stability of the considered scheme. We perturb the initial values $\tilde{y}_i^0 = \tilde{u}_0(x_i^0) = u_0(x_i^n) + E_S \cos(100 x_i^0 \pi / l),\ i \in P_0$ (see Fig. 2).

Table 2. $l = 2,\ T = 1$, $\beta_0 = 0.5$, $\epsilon = 1.0 \cdot 10^{-15}$, $h_i^0 = 0.02$, $\tau = 0.001$, $m = 10$.

E_S	$\max\limits_{0 \le n \le N_0} \lVert \tilde{y}^n - y \rVert_{\overline{C}_n}$
10^0	$1.23 \cdot 10^{+01}$
10^{-1}	$8.54 \cdot 10^{-01}$
10^{-2}	$8.20 \cdot 10^{-02}$
10^{-5}	$8.16 \cdot 10^{-05}$
10^{-10}	$8.16 \cdot 10^{-10}$

Fig. 2. The exact solution and $\tilde{y}_i^0$ for $l = 2,\ T = 1$, $\beta_0 = 0.5$, $E_S = 10^{-1}$, $h_i^0 = 0.02$.

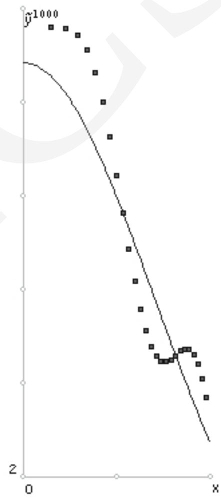

Fig. 3. The exact solution and $\tilde{y}_i^{1000}$ for $l = 2,\ T = 1$, $\beta_0 = 0.5$, $E_S = 10^{-1}$, $h_i^0 = 0.02$.

To obtain better numerical results, under the condition $f(u) \in C^{2M+2}(\mathbb{R})$, where $M = const > 0$, we use the Euler-Maclaurin formula instead of the trapezoidal rule [2]

$$\int\limits_{u_i^n}^{u_i^{n+1}} \frac{dx}{f(u)} \approx \frac{y_i^{n+1} - y_i^n}{m} \left(\frac{1}{2f\left(y_i^{n+1}\right)} + \sum_{j=1}^{m-1} \frac{1}{f\left(y_i^n + j\frac{y_i^{n+1} - y_i^n}{m}\right)} + \frac{1}{2f\left(y_i^n\right)} \right)$$
$$+ \sum_{j=1}^{M} (-1)^j a_j \left(\frac{y_i^{n+1} - y_i^n}{m} \right)^{2j} \left[\left(\frac{1}{f\left(y_i^{n+1}\right)} \right)^{(2j-1)} - \left(\frac{1}{f\left(y_i^n\right)} \right)^{(2j-1)} \right],$$

where a_j is calculated from $\frac{1}{2M+1} = \frac{1}{2} + \sum_{j=1}^{M} (-1)^j \frac{(2M)!}{(2M-2j+1)!} a_j$. We estimate a posteriori the error of this method [13]

$$p^N = \log_2 \left(D^N / D^{2N} \right),$$

where

$$D^N = \frac{1}{2^{2M+2}-1} \max_{n=0,N_0} \left\| y^n - y^{2n} \right\|_{\overline{C}_n},\ x_i^n \in \overline{\omega},\ x_i^{2n} \in \overline{\omega}_{h\tau/2},$$

$$D^{2N} = \frac{1}{2^{2M+2}-1} \max_{n=0,2N_0} \left\| y^n - y^{2n} \right\|_{\overline{C}_n},\ x_i^n \in \overline{\omega}_{h\tau/2},\ x_i^{2n} \in \overline{\omega}_{h\tau/4}.$$

Here we use two additional grids $\overline{\omega}_{h\tau/2}$, $\overline{\omega}_{h\tau/4}$ with the time step twice and four times smaller than τ, respectively. Tables 3 - 4 present numerically that the error $\max\limits_{0 \le n \le N_0} \| y^n - u^n \|_{\overline{C}_n}$ of the method applying the Euler-Macluarin formula equals $O\left(\left(\frac{\tau}{m}\right)^{2M+2} \right)$.

Table 3. $\beta_0 = 0.5,\ l = 2,\ T = 1, M = 1, \epsilon = 1.0 \cdot 10^{-19}, h_i^0 = 0.2, m = 1.$

τ	$\max\limits_{0\le n\le N_0} \|\|y^n - u^n\|\|_{\overline{C}_n}$	D^N	p^N
0.5	$6.67 \cdot 10^{-02}$	–	–
0.25	$3.43 \cdot 10^{-03}$	$2.14 \cdot 10^{-04}$	–
0.125	$3.16 \cdot 10^{-04}$	$1.98 \cdot 10^{-05}$	3.525
0.0625	$2.75 \cdot 10^{-05}$	$1.72 \cdot 10^{-06}$	4.001
0.03125	$1.71 \cdot 10^{-06}$	$1.07 \cdot 10^{-07}$	4.000

Table 4. $\beta_0 = 0.5,\ l = 2,\ T = 1, M = 2, \epsilon = 1.0 \cdot 10^{-19}, h_i^0 = 0.2, m = 1.$

τ	$\max\limits_{0\le n\le N_0} \|\|y^n - u^n\|\|_{\overline{C}_n}$	D^N	p^N
0.5	$2.80 \cdot 10^{-02}$	–	–
0.25	$4.32 \cdot 10^{-04}$	$6.75 \cdot 10^{-06}$	–
0.125	$1.02 \cdot 10^{-05}$	$1.59 \cdot 10^{-07}$	5.457
0.0625	$2.32 \cdot 10^{-07}$	$6.32 \cdot 10^{-09}$	6.003
0.03125	$3.61 \cdot 10^{-09}$	$5.65 \cdot 10^{-11}$	6.001

9 Conclusions

In the paper, we have considered the EDS for the Cauchy problem for the semilinear parabolic equation. The solution with the separated variables $u(x,t) = X(x)T_1(t) + T_2(t)$ was a very important assumption. The ADSs have been constructed in the case when integrals in EDS cannot be evaluated exactly. The special attention was paid to investigate approximation, stability and convergence of the nonlinear scheme. Convergence of the iteration method was also considered. Numerical results have been presented to confirm the theoretical results given in the paper.

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